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Reasoning in a rational extension of SROEL

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Reasoning in a Rational Extension of $\mathcal{SROEL}(\sqcap, \times)$

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Abstract. In this work we define a rational extension $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ of the low complexity description logic $\mathcal{SROEL}(\sqcap, \times)$, which underlies the OWL EL ontology language. The logic is extended with a typicality operator $\mathbf{T}$, whose semantics is based on Lehmann and Magidor's ranked models and allows for the definition of defeasible inclusions. We develop a Datalog materialization calculus for rational entailment in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$, showing that instance checking can be computed in polynomial time. Also, we show that deciding instance checking under minimal entailment is a coNP-hard problem. An extended abstract of this paper appears in [22].

1 Introduction

The need for extending Description Logics (DLs) with nonmonotonic features has led, in the last decade, to the development of several extensions of DLs, obtained by combining them with the most well-known formalisms for nonmonotonic reasoning [31, 2, 11, 13, 16, 21, 20, 24, 8, 6, 10, 30, 25, 9, 3] to deal with defeasible reasoning and inheritance, to allow for prototypical properties of concepts and to combine DLs with nonmonotonic rule-based languages under the answer set semantics [13], the well-founded semantics [12], the MKNF semantics [25], as well as in Datalog +/- [23]. Systems integrating Answer Set Programming (ASP) [14] and DLs have been developed, e.g., the DReW System for Nonmonotonic DL-Programs [32].

In this paper we study a preferential extension of the logic $\mathcal{SROEL}(\sqcap, \times)$, introduced by Krötzsch [27], which is a low-complexity description logic of the $\mathcal{EL}$ family [1] that includes local reflexivity, conjunction of roles and concept products and is at the basis of OWL 2 EL. Our extension is based on Kraus, Lehmann and Magidor (KLM) preferential semantics [26], and, specifically, on ranked models [29]. We call the logic $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ and define notions of rational and minimal entailment for it.

The semantics of ranked interpretations for DLs was first studied in [8], where a rational extension of $\mathcal{ALC}$ is developed allowing for defeasible concept inclusions of the form $C \mathrel{\tilde{\sqsubseteq}} D$. In this work, following [17, 21], we extend the language of $\mathcal{SROEL}(\sqcap, \times)$ with typicality concepts of the form $\mathbf{T}(C)$, whose instances are intended to be the typical C elements. Typicality concepts can be used to express defeasible inclusions of the form $\mathbf{T}(C) \sqsubseteq D$ ("the typical C elements are D"). Here, however, as in [7, 15], we allow for typicality concepts to freely occur in concept inclusions. In this respect, the language with typicality that we consider is more general than the language with typicality in [21], where minimal ranked models have been shown to provide a semantic characterization to rational closure for the description logic $\mathcal{ALC}$, generalizing to

DLs the rational closure by Lehmann and Magidor [29]. Alternative constructions of rational closure for $\mathcal{ALC}$ have been proposed in [10, 9]. All such constructions regard languages only containing strict or defeasible inclusions. In particular, in the language with typicality, $\mathbf{T}(C)$ may occur in inclusions $\mathbf{T}(C) \sqsubseteq D$ (but also in assertions).

In this work, we define a Datalog translation for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ which builds on the materialization calculus in [27], and, for typicality reasoning, is based on properties of ranked models, showing that instance checking for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ can be computed in polynomial time under the rational entailment. While this result has the consequence that the Rational Closure for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ (based on definition in [21]) can be computed in polynomial time, we show that, for general $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ KBs, deciding instance checking under minimal entailment is a CONP-hard problem.

2 A rational extension of $\mathcal{SROEL}(\sqcap, \times)$

In this section we extend the notion of concept in $\mathcal{SROEL}(\sqcap, \times)$ adding typicality concepts (we refer to [27] for a detailed description of the syntax and semantics of $\mathcal{SROEL}(\sqcap, \times)$). We let N_C be a set of concept names, N_R a set of role names and N_I a set of individual names. A concept in $\mathcal{SROEL}(\sqcap, \times)$ is defined as follows:

$$C := A \mid \top \mid \bot \mid C \sqcap C \mid \exists r.C \mid \exists S.Self \mid \{a\}$$

where $A \in N_C$ and $r \in N_R$. We introduce a notion of *extended concept* C_E as follows:

$$C_E := C \mid \mathbf{T}(C) \mid C_E \sqcap C_E \mid \exists S.C_E$$

where C is a $\mathcal{SROEL}(\sqcap, \times)$ concept. Hence, any concept of $\mathcal{SROEL}(\sqcap, \times)$ is also an extended concept; a typicality concept $\mathbf{T}(C)$ is an extended concept and can occur in conjunctions and existential restrictions, but it cannot be nested.

A KB is a triple $(TBox, RBox, ABox)$. $TBox$ contains a finite set of *general concept inclusions* (GCI) $C \sqsubseteq D$, where C and D are extended concepts; $RBox$ (as in [27]) contains a finite set of *role inclusions* of the form $S \sqsubseteq T$, $R \circ S \sqsubseteq T$, $S_1 \sqcap S_2 \sqsubseteq T$, $R \sqsubseteq C \times D$, where C and D are concepts, $R, S, T \in N_R$. $ABox$ contains *individual assertions* of the form $C(a)$ and $R(a, b)$, where $a, b \in N_I$, $R \in N_R$ and C is an extended concept. Restrictions are imposed on the use of roles as in [27].

Following [8, 21], a semantics for the extended language is defined, adding to interpretations in $\mathcal{SROEL}(\sqcap, \times)$ [27] a *preference relation* $<$ on the domain, which is intended to compare the "typicality" of domain elements. The typical instances of a concept C, i.e., the instances of $\mathbf{T}(C)$, are the instances x of C that are minimal with respect to $<$.

Definition 1. *A $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ interpretation $\mathcal{M}$ is any structure $\langle \Delta, <, \cdot^I \rangle$ where:*

- *Δ is a domain; $\cdot^I$ is an interpretation function that maps each concept name A to set $A^I \subseteq \Delta^I$, each role name r to a binary relation $r^I \subseteq \Delta^I \times \Delta^I$, and each individual name a to an element $a^I \in \Delta^I$. $\cdot^I$ is extended to complex concepts as usual:*

$\top^I = \Delta$; $\quad \bot^I = \emptyset$; $\quad \{a\}^I = \{a^I\}$;
$(C \sqcap D)^I = C^I \cap D^I$;
$(\exists r.C)^I = \{x \in \Delta \mid \exists y \in C^I : (x, y) \in r^I\}$;
$(\exists R.Self)^I = \{x \in \Delta \mid (x, x) \in R^I\}$.

– *$<$ is an irreflexive, transitive, well-founded and modular relation over Δ;*
– *the interpretation of concept $\mathbf{T}(C)$ is defined as follows:*

$$(\mathbf{T}(C))^I = Min_<(C^I)$$

where $Min_<(S) = \{u : u \in S$ and $\nexists z \in S$ s.t. $z < u\}$.

Furthermore, an irreflexive and transitive relation $<$ is well-founded *if, for all $S \subseteq \Delta$, for all $x \in S$, either $x \in Min_<(S)$ or $\exists y \in Min_<(S)$ such that $y < x$. It is* modular *if, for all $x, y, z \in \Delta$, $x < y$ implies $x < z$ or $z < y$.*

As in [29], modularity in preferential models can be equivalently defined by postulating the existence of a rank function $k_{\mathcal{M}} : \Delta \longmapsto \Omega$, where Ω is a totally ordered set. Hence, modular preferential models are called *ranked models*. The preference relation $<$ can be defined from $k_{\mathcal{M}}$ as follows: $x < y$ if and only if $k_{\mathcal{M}}(x) < k_{\mathcal{M}}(y)$.

Given an interpretation $\mathcal{M}$ the notions of satisfiability and entailment are defined as usual.

Definition 2 (Satisfiability and rational entailment). *An interpretation $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ satisfies:*
- *a concept inclusion $C \sqsubseteq D$ if $C^I \subseteq D^I$;*
- *a role inclusion $S \sqsubseteq T$ if $S^I \subseteq T^I$;*
- *a generalized role inclusion $R \circ S \sqsubseteq T$ if $R^I \circ S^I \subseteq T^I$*
(where $R^I \circ S^I = \{(x, z) \mid (x, y) \in R^I$ and $(y, z) \in S^I$, for some $y \in \Delta\}$);
- *a role conjunction $S_1 \sqcap S_2 \sqsubseteq T$ if $S_1^I \cap S_2^I \subseteq T^I$;*
- *a concept product axiom $C \times D \sqsubseteq T$ if $C^I \times D^I \subseteq T^I$;*
- *a concept product axiom $R \sqsubseteq C \times D$ if $R^I \subseteq C^I \times D^I$;*
- *an assertion $C(a)$ if $a^I \in C^I$;*
- *an assertion $R(a, b)$ if $(a^I, b^I) \in R^I$.*

Given a KB $K = (TBox, RBox, ABox)$, an interpretation $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ satisfies *$TBox$ (resp., $RBox$, $ABox$) if $\mathcal{M}$ satisfies all axioms in $TBox$ (resp., $RBox$, $ABox$), and we write $\mathcal{M} \models TBox$ (resp., $RBox$, $ABox$). An interpretation $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ is a* model *of K (and we write $\mathcal{M} \models K$) if $\mathcal{M}$ satisfies all the axioms in $TBox$, $RBox$ and $ABox$.*

Let a query F be either a concept inclusion $C \sqsubseteq D$, where C and D are extended concepts, or an individual assertion. F is rationally entailed by *K, written $K \models_{sroelrt} F$, if for all models $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ of K, $\mathcal{M}$ satisfies F.*

Consider the following example of knowledge base, stating that: typical Italians have black hair; typical students are young; they hate math, unless they are nerd (in which case they love math); all Mary's friends are typical students. We also have the assertions stating that Mary is a student, that Mario is an Italian student, and is a friend of Mary, Luigi is a typical Italian student, and Paul is a typical young student.

Example 1. $TBox$:
(a) $\mathbf{T}(\mathit{Italian}) \sqsubseteq \exists \mathit{hasHair}.\{\mathit{Black}\}$
(b) $\mathbf{T}(\mathit{Student}) \sqsubseteq \mathit{Young}$
(c) $\mathbf{T}(\mathit{Student}) \sqsubseteq \mathit{MathHater}$
(d) $\exists \mathit{hasHair}.\{\mathit{Black}\} \sqcap \exists \mathit{hasHair}.\{\mathit{Blond}\} \sqsubseteq \bot$
(e) $\mathit{MathLover} \sqcap \mathit{MathHater} \sqsubseteq \bot$
(f) $\exists \mathit{friendOf}.\{\mathit{mary}\} \sqsubseteq \mathbf{T}(\mathit{Student})$

$ABox$: $\mathit{Student}(\mathit{mary})$, $\mathit{friendOf}(\mathit{mario}, \mathit{mary})$, $(\mathit{Student} \sqcap \mathit{Italian})(\mathit{mario})$, $\mathbf{T}(\mathit{Student} \sqcap \mathit{Italian})(\mathit{luigi})$, $\mathbf{T}(\mathit{Student} \sqcap \mathit{Young})(\mathit{paul})$

The fact that concepts $\mathbf{T}(C)$ can occur anywhere (apart from being nested in a $\mathbf{T}$ operator) can be used, e.g., to state that typical working students inherit properties of typical students ($\mathbf{T}(\mathit{Student} \sqcap \mathit{Worker}) \sqsubseteq \mathbf{T}(\mathit{Student})$), in a situation in which typical students and typical workers have conflicting properties (e.g., as regards paying taxes). Also, we could state that there are typical students who are Italian: $\top \sqsubseteq \exists U.\mathbf{T}(\mathit{Student} \sqcap \mathit{Italian})$, where U is the universal role ($\top \times \top \sqsubseteq U$).

Standard DL inferences hold for $\mathbf{T}(C)$ concepts and $\mathbf{T}(C) \sqsubseteq D$ inclusions. For instance, we can conclude that Mario is a a typical student (by (f)) and young (by (b)). However, by the properties of defeasible inclusions, Luigi, who is a typical Italian student, and Paul, who is a typical young student, both inherit the property of typical students of being math haters (respectively, by rational monotonicity and by cautious monotonicity).

A normal form for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ knowledge bases can be defined. A KB in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ is in *normal form* if it admits all the axioms of a $\mathcal{SROEL}(\sqcap, \times)$ KB in normal form:

$$\begin{array}{c} C(a) \qquad R(a,b) \qquad A \sqsubseteq \bot \qquad \top \sqsubseteq C \qquad A \sqsubseteq \{c\} \\ A \sqsubseteq C \qquad A \sqcap B \sqsubseteq C \qquad \exists R.A \sqsubseteq C \qquad A \sqsubseteq \exists R.B \\ \{a\} \sqsubseteq C \qquad \exists R.\mathit{Self} \sqsubseteq C \qquad A \sqsubseteq \exists R.\mathit{Self} \\ R \sqsubseteq T \qquad R \circ S \sqsubseteq T \qquad R \sqcap S \sqsubseteq T \qquad A \times B \sqsubseteq R \qquad R \sqsubseteq C \times D \end{array}$$

(where $A, B, C, D \in N_C$, $R, S, T \in N_R$ and $a, b, c \in N_I$) and, in addition, it admits axioms of the form: $A \sqsubseteq T(B)$ and $T(B) \sqsubseteq C$ with $A, B, C \in N_C$. Extending the results in [1] and in [27], it is easy to see that, given a $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ KB, a semantically equivalent KB in normal form (over an extended signature) can be computed in linear time. In essence, for each concept $\mathbf{T}(C)$ occurring in the KB, we introduce two new concept names, X_C and Y_C. A new KB is obtained by replacing all the occurrences of $\mathbf{T}(C)$ with X_C in all the inclusions and assertions, and adding the following additional inclusion axioms:

$$X_C \sqsubseteq \mathbf{T}(Y_C), \quad \mathbf{T}(Y_C) \sqsubseteq X_C, \quad Y_C \sqsubseteq C, \quad C \sqsubseteq Y_C$$

Then the new KB undergoes the normal form transformation for $\mathcal{SROEL}(\sqcap, \times)$ [27]. The resulting KB is linear in the size of the original one.

3 Minimal entailment

In Example 1, we cannot conclude that all typical young Italians have black hair (and that Luigi has black hair) using rational monotonicity, as we do not know whether there is some typical Italian who is young. To supports such a stronger nonmonotonic inference, a minimal model semantics is needed to select those interpretation where individuals are as typical as possible.

We consider the notion of minimal canonical model in [21]. Given a KB K and a query F, let $\mathcal{S}$ be the set of all the concepts (and subconcepts) occurring in K or F together with their complements ($\mathcal{S}$ is finite).

Definition 3 (Canonical models). *A model $\mathcal{M} = \langle \Delta, <, I \rangle$ of K is* canonical *if, for each set of $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ concepts $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{S}$ consistent with K (i.e., s.t. $K \not\models_{sroelrt} C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n \sqsubseteq \bot$), there exists (at least) a domain element $x \in \Delta$ such that $x \in (C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n)^I$.*

Among canonical models, we select the minimal ones according to the following *preference relation $\prec$ over the set of ranked interpretations*. An interpretation $\mathcal{M} = \langle \Delta, <, I \rangle$ *is preferred to* $\mathcal{M}' = \langle \Delta', <', I' \rangle$ ($\mathcal{M} \prec \mathcal{M}'$) if: $\Delta = \Delta'$; $C^I = C^{I'}$ for all concepts C; for all $x \in \Delta$, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$, and there exists $y \in \Delta$ such that $k_{\mathcal{M}}(y) < k_{\mathcal{M}'}(y)$.

Definition 4. *$\mathcal{M}$ is a* minimal canonical model *of K if it is a canonical model of K and it is minimal among all the canonical models of K wrt. the preference relation $\prec$.*

Definition 5 (Minimal entailment). *Given a query F, F* is minimally entailed by *K, written $K \models_{min} F$ if, for all minimal canonical models $\mathcal{M}$ of K, $\mathcal{M}$ satisfies F.*

We can see that, under minimal entailment, from the KB in Example 1 we can conclude $\exists hasHair.\{Black\}(luigi)$ and $\mathbf{T}(Young \sqcap Italian) \sqsubseteq \exists hasHair.\{Black\}$, as nothing prevents a $Young \sqcap Italian$ individual from having rank 0.

We can show that the problem of instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ under minimal entailment is CONP-hard. The proof is based on a reduction from tautology checking of propositional 3DNF formulae to instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$.

Theorem 1. *Instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ under minimal entailment is* CONP-*hard.*

Proof. ***(sketch)*** Given an alphabet of propositional variables $L = \{p_1, \ldots, p_k\}$, let $\gamma = G_1 \vee \ldots \vee G_n$ be a propositional formula in 3DNF, where, for each $i = 1, \ldots, n$, $G_i = l_i^1 \wedge l_i^2 \wedge l_i^3$ and each l_i^j ($j = 1, \ldots, 3$) is a literal (a variable $p \in L$ or its negation $\neg p$).

We define a KB $K = (TBox, RBox, ABox)$ in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ as follows. We introduce two concept names $P_h, \overline{P}_h \in N_C$ for each variable $p_h \in L$. Also, we introduce in N_C a concept name D_γ associated with the formula γ, a concept name D_i associated with disjunct G_i, for each $i = 1, \ldots, n$, and a role name R. We let $a \in N_I$ be an individual name, and we define K as follows: $RBox = \{P_h \times \overline{P}_h \sqsubseteq R, h = 1, ..., k\}$, $ABox = \{\mathbf{T}(P_h \sqcap \overline{P}_h)(a), h = 1, ..., k\}$, and $TBox$ contains the following inclusions:

(1) $\mathbf{T}(P_h) \sqcap \mathbf{T}(\overline{P}_h) \sqsubseteq \bot$,

(2) $\mathbf{T}(\top) \sqsubseteq A \qquad A \sqcap B \sqsubseteq \bot \qquad \exists R.A \sqsubseteq B$
(3) $D_i \sqsubseteq D_\gamma$
(4) $C_i^1 \sqcap C_i^2 \sqcap C_i^3 \sqsubseteq D_i$
(5) $D_i \sqsubseteq C_i^1 \sqcap C_i^2 \sqcap C_i^3$
for each $h = 1, \ldots, k$ and for each $i = 1, \ldots, n$, where U is the universal role and (for $j \in \{1, 2, 3\}$) C_i^j is

$$C_i^j = \begin{cases} \mathbf{T}(P_h) & \text{if } l_i^j = p_h \\ \exists U.(\mathbf{T}(\top) \sqcap \mathbf{T}(P_h)) & \text{if } l_i^j = \neg p_h \end{cases}$$

Let us consider any model $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ of K. Observe that, as $a^I \in P_h \sqcap \overline{P}_h$, a^I cannot have rank 0, otherwise it would be both a typical P_h and a typical $\overline{P}_h$, falsifying (1). By the role inclusions each P_h element is in relation R with a $\overline{P}_h$ element. Also, by (2), there cannot be both a P_h element and a $\overline{P}_h$ element with rank 0, otherwise they would be both A elements related by R (which is excluded by axioms on line (2)). It is possible that neither P_h elements nor $\overline{P}_h$ elements have rank 0. In minimal canonical models of K, however, there must be either a P_h element or a $\overline{P}_h$ element with rank 0.

Let $k_{\mathcal{M}}(C) = min\{k_{\mathcal{M}}(x) \mid x \in C^I\}$ be the rank of a concept C in a ranked model $\mathcal{M}$. It can be seen that, in all the minimal canonical models of K, for all $h = 1, \ldots, k$:

- either $k_{\mathcal{M}}(P_h) = 0$ or $k_{\mathcal{M}}(\overline{P}_h) = 0$;
- $k_{\mathcal{M}}(P_h \sqcap \overline{P}_h) = 1$ and $k_{\mathcal{M}}(a^I) = 1$.

As a consequence, a^I is either a typical P_h element (when the rank of $\overline{P}_h$ is 0) or a typical $\overline{P}_h$ (when the rank of P_h is 0). So there are alternative minimal canonical models in which, for each h, a^I is either a $\mathbf{T}(P_h)$, and in this case there exists a typical $\overline{P}_h$ element with rank 0; or a is a $\mathbf{T}(\overline{P}_h)$, and in this case there exists a typical P_h element with rank 0. Therefore, in any minimal canonical models $\mathcal{M}$ of K: either $a^I \in (\mathbf{T}(P_h))^I$ or $a^I \in (\exists U.(\mathbf{T}(\top) \sqcap \mathbf{T}(P_h)))^I$ (but not both). Hence, for a^I the two concepts in the definition of C_i^j are disjoint and complementary. Then the following holds:

$K \models_{min} D_\gamma(a)$ if and only if γ is a tautology $\square$

Observe that, for KBs which only allow typicality concepts to occur on the left hand side of typicality inclusions (call them *simple* KBs), Theorem 1 does not provide a lower bound for minimal entailment. In fact, inclusion (5) may contain the typicality operator on the right hand side. It is open whether a similar proof can be done also for simple knowledge bases in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$. For simple KBs in the rational extension of $\mathcal{ALC}$, it was proved in [21] that all minimal canonical models of the KB assign the same ranks to concepts, namely, the ranks determined by the rational closure construction. Clearly, this must be also true for the fragment of $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ included in the language of $\mathcal{ALC}$. Note that K, in the proof above, has alternative minimal canonical models with incomparable rank assignments. The existence of alternative minimal models for a KB with free occurrences of typicality in the propositional case was observed in [7] for Propositional Typicality logic (PTL) (see Example 3 therein, which, however, exploits negation and hence it is not in the language of $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$.

4 Deciding rational entailment in polynomial time

While instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ under minimal entailment is CONP-hard, in this section we prove that instance checking under rational entailment can be decided in polynomial time for normalized KBs, by defining a translation of a normalized KB into a set Datalog rules, whose grounding is polynomial in the size of the KB. In particular, we extend the Datalog materialization calculus for $\mathcal{SROEL}(\sqcap, \times)$, proposed by Krötzsch [27], to deal with typicality concepts and with instance checking under rational entailment in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$.

The calculus in [27] uses predicates $inst(a, C)$ (whose meaning includes: the individual a is an instance of concept name C, see [28] for details), $triple(a, R, b)$ (a is in relation R with b), $self(a, R)$ (a is in relation R with itself). In order to map a $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ KB to a Datalog program, we add predicates to represent that: an individual a is a typical instance of a concept name ($typ(a, C)$); the ranks of two individuals a and b are the same ($sameRank(a, b)$); the rank of a is less or equal than the one of b ($leqRank(a, b)$).

Besides the constants for individuals in N_I (which are assumed to be finitely many), the calculus in [27] exploits auxiliary constants $aux^{A \sqsubseteq \exists R.C}$ (one for each inclusion of the form $A \sqsubseteq \exists R.C$) to deal with existential restriction. We also need to introduce an auxiliary constant aux_C for any concept $\mathbf{T}(C)$ occurring in the KB or in the query, used as a representative typical C, in case C is non-empty.

Given a normalized KB $K = (TBox, RBox, ABox)$ and query Q of the form $C(a)$ or $\mathbf{T}(C)(a)$, where C is a concept in the normalized KB, the Datalog program for instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$, i.e. for querying whether$K \models_{sroelrt} Q$, is a program $\Pi(K)$, the union of:

1. Π_K, the representation of K as a set of Datalog facts, based on the input translation in [27];
2. Π_{IR}, the inference rules of the basic calculus in [27];
3. Π_{RT}, containing the rules for reasoning with typicality in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$.

A query Q of the form $\mathbf{T}(C)(a)$, or $C(a)$, is mapped to a goal G_Q of the form $typ(a, C)$, or $inst(a, C)$. We define $\Pi(K)$ in such a way that G_Q is derivable in Datalog from $\Pi(K)$ (written $\Pi(K) \vdash G_Q$) if and only if $K \models_{sroelrt} Q$.

Π_K includes the result of the input translation in section 3 in [27] where $nom(a)$, $cls(A)$, $rol(R)$ are used for $a \in N_I$, $A \in N_C$, $R \in N_R$, and, for example:

- $subClass(a, C)$, $subClass(A, c)$, $subClass(A, C)$ are used for $C(a)$, $A \sqsubseteq \{c\}$, $A \sqsubseteq C$;
- $subEx(R, A, C)$ is used for $\exists R.A \sqsubseteq C$;

and similar statements represent other axioms in the normalized KB.

The following is the additional mapping for the extended syntax of the normal form for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ (note that no mapping is needed for assertions $\mathbf{T}(C)(a)$, as they do not occur in a normalized KB):

$$A \sqsubseteq T(B) \mapsto supTyp(A, B)$$
$$T(B) \sqsubseteq C \mapsto subTyp(B, C)$$

Also, we need to add $top(\top)$ to the input specification.

Π_{IR} contains all the inference rules from [27][1]:

(1) $inst(x, x) \leftarrow nom(x)$
(2) $self(x, v) \leftarrow nom(x), triple(x, v, x)$
(3) $inst(x, z) \leftarrow top(z), inst(x, z')$
(4) $inst(x, y) \leftarrow bot(z), inst(u, z), inst(x, z'), cls(y)$
(5) $inst(x, z) \leftarrow subClass(y, z), inst(x, y)$
(6) $inst(x, z) \leftarrow subConj(y1, y2, z), inst(x, y1), inst(x, y2)$
(7) $inst(x, z) \leftarrow subEx(v, y, z), triple(x, v, x'), inst(x', y)$
(8) $inst(x, z) \leftarrow subEx(v, y, z), self(x, v), inst(x, y)$
(9) $triple(x, v, x') \leftarrow supEx(y, v, z, x'), inst(x, y)$
(10) $inst(x', z) \leftarrow supEx(y, v, z, x'), inst(x, y)$
(11) $inst(x, z) \leftarrow subSelf(v, z), self(x, v)$
(12) $self(x, v) \leftarrow supSelf(y, v), inst(x, y)$
(13) $triple(x, w, x') \leftarrow subRole(v, w), triple(x, v, x')$
(14) $self(x, w) \leftarrow subRole(v, w), self(x, v)$
(15) $triple(x, w, x'') \leftarrow subRChain(u, v, w), triple(x, u, x'), triple(x', v, x'')$
(16) $triple(x, w, x') \leftarrow subRChain(u, v, w), self(x, u), triple(x, v, x')$
(17) $triple(x, w, x') \leftarrow subRChain(u, v, w), triple(x, u, x'), self(x', v)$
(18) $triple(x, w, x) \leftarrow subRChain(u, v, w), self(x, u), self(x, v)$
(19) $triple(x, w, x') \leftarrow subRConj(v1, v2, w), triple(x, v1, x'), triple(x, v2, x')$
(20) $self(x, w) \leftarrow subRConj(v1, v2, w), self(x, v1), self(x, v2)$
(21) $triple(x, w, x') \leftarrow subProd(y1, y2, w), inst(x, y1), inst(x', y2)$
(22) $self(x, w) \leftarrow subProd(y1, y2, w), inst(x, y1), inst(x, y2)$
(23) $inst(x, z1) \leftarrow supProd(v, z1, z2), triple(x, v, x')$
(24) $inst(x, z1) \leftarrow supProd(v, z1, z2), self(x, v)$
(25) $inst(x', z2) \leftarrow supProd(v, z1, z2), triple(x, v, x')$
(26) $inst(x, z2) \leftarrow supProd(v, z1, z2), self(x, v)$
(27) $inst(y, z) \leftarrow inst(x, y), nom(y), inst(x, z)$
(28) $inst(x, z) \leftarrow inst(x, y), nom(y), inst(y, z)$
(29) $triple(z, u, y) \leftarrow inst(x, y), nom(y), triple(z, u, x)$

Note that $inst(c, d)$ for $c, d \in N_I$ means [28] that $\{c\} \sqsubseteq \{d\}$, i.e., c and d represent the same domain element.

Π_{RT}, i.e. the set of rules to deal with typicality, is as follows; it contains rules for $supTyp$ and $subTyp$ axioms, and rules that deal with the rank of domain elements. In the rules, x, y, z, A, B, C are all Datalog variables.

$(SupTyp)$ $typ(x, z) \leftarrow supTyp(y, z), inst(x, y)$
$(SubTyp)$ $inst(x, z) \leftarrow subTyp(y, z), typ(x, y)$
$(Refl)$ $inst(x, y) \leftarrow typ(x, y)$
$(A0)$ $typ(aux_C, C) \leftarrow inst(x, C)$
$(A1)$ $leqRank(x, y) \leftarrow typ(x, B), inst(y, B)$

[1] Here, u, v, x, y, z, w, possibly with suffixes, are variables.

$(A2)$ $sameRank(x, y) \leftarrow typ(x, A), typ(y, A)$
$(A3)$ $typ(x, B) \leftarrow sameRank(x, y), inst(x, B), typ(y, B)$
$(A4)$ $typ(x, B) \leftarrow inst(x, A), supTyp(A, B)$
$(B1)$ $sameRank(x, z) \leftarrow sameRank(x, y), sameRank(y, z)$
$(B2)$ $sameRank(x, y) \leftarrow sameRank(y, x)$
$(B3)$ $sameRank(x, x) \leftarrow \top$
$(B4)$ $leqRank(x, y) \leftarrow sameRank(y, x)$
$(B5)$ $leqRank(x, z) \leftarrow leqRank(x, y), leqRank(y, z)$
$(B6)$ $sameRank(x, y) \leftarrow leqRank(x, y), leqRank(y, x)$
$(B7)$ $sameRank(x, y) \leftarrow nom(y), inst(x, y)$

Rule $(Refl)$ corresponds to the reflexivity postulate. Rules $(A0)-(A4)$ encode properties of ranked models: if there is a C element, there must be a typical C element $(A0)$; a typical B element has a rank less or equal to the rank of any B element $(A1)$; two elements which are both typical A elements have the same rank $(A2)$; if x is a B element and has the same rank as a typical B element, x is also a typical B element $(A3)$; if x is an A element and all A's are typical B's, then x is a typical A $(A4)$. $(B1) - (B7)$ define properties of rank order. In particular, by $(B7)$, two constants that correspond to the same domain element have the same rank.

The postulates of rational consequence relation are enforced by the specification above. Consider, for instance, (CM). Suppose that $subTyp(A, B)$ and $subTyp(A, C)$ are in Π_K (as $\mathbf{T}(A) \sqsubseteq B$, $\mathbf{T}(A) \sqsubseteq C$ are in K) and that D is a concept name defined to be equivalent to $A \sqcap B$ in K. Suppose that $typ(a, D)$ holds. One can infer $typ(a, A)$ and hence $inst(a, C)$, i.e., typical $A \sqcap B$'s inherit from typical A's the property of being C's (the inference for $Paul$ in Example 1). In fact, $typ(a, A)$ is inferred showing that a (who is a typical D and an A, as it is a D) and aux_A (who is a typical A, by $(A1)$, and a D, since all the typical A's are also B's and hence $A \sqcap B$'s) have the same rank. In fact, using$(A1)$ twice, one can conclude both $leqRank(a, aux_A)$ and $leqRank(aux_A, a)$ so that, by $(B6)$, $sameRank(a, aux_A)$. Then, by $(A3)$, we infer $typ(a, A)$. With rule $(subTyp)$, from $typ(a, A)$ and $subTyp(A, C)$, we conclude $inst(a, C)$. Reasoning in a similar way, one can see that also (RM) and (LLE) are enforced by the rules above. Inference in $\mathcal{SROEL}(\sqcap, \times)$ already deals with conjunctive consequences (And) and right weakening (RW).

Theorem 2. *For a $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ KB in normal form K, and a query Q of the form $\mathbf{T}(C)(a)$ or $C(a)$, $K \models_{sroelrt} Q$ if and only if $\Pi(K) \vdash G_Q$.*

$\Pi(K)$ contains a polynomial number of rules and exploits a polynomial number of concepts in the size of K, hence instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ can be decided in polynomial time using the calculus in Datalog. The encoding can be processed, e.g., in an ASP solver such as Clingo or DLV (with the proper capitalization of variables); computation of the (unique, in this case) answer set takes a negligible time for KBs with a hundred assertions (half of them with $\mathbf{T}$).

Exploiting the approach presented in [27], a version of the Datalog specification where predicates have an additional parameter (and is therefore less efficient than the previous one) can be used to check subsumption for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$. This also provides a polynomial upper bound for reasoning under rational closure in $\mathcal{SROEL}(\sqcap, \times)$.

In fact, the rational closure of a simple KB can be defined based on the same construction given in [21] for $\mathcal{ALC}$, by exploiting an iterative algorithm which verifies the exceptionality of concepts by subsumption checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$. In particular, one can define, for a simple KB K the notion of exceptionality as follows: C *is exceptional wrt* K iff $K \models_{sroelrt} \mathbf{T}(\top) \sqcap C \sqsubseteq \bot$. This subsumption is not in the language of normalized KBs, but it can be replaced by $A \sqsubseteq \bot$, adding $\mathbf{T}(\top) \sqsubseteq X$ and $X \sqcap C \sqsubseteq A$ to K. The construction requires a quadratic number of subsumption checks.

5 Conclusions and Related Work

In this paper we defined a rational extension $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\,\mathbf{T}$ of the low complexity description logic $\mathcal{SROEL}(\sqcap, \times)$, which underlies the OWL EL ontology language, introducing a typicality operator. Building on the materialization calculus for $\mathcal{SROEL}(\sqcap, \times)$ in Datalog presented in [27], a calculus for instance checking and subsumption under rational entailment is defined, showing that these problems can be decided in polynomial time. This result also provides a polynomial upper bound for reasoning under rational closure in $\mathcal{SROEL}(\sqcap, \times)$.

For general KBs, we have shown that minimal entailment in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ is coNP-hard. When free occurrences of typicality concepts in concept inclusions are allowed, alternative minimal models may exist with different rank assignments to concepts. In [7] this phenomenon has been analyzed in the context of PTL, considering alternative preference relations over ranked interpretations which coincide over simple KBs but, for general ones, define different notions of entailment satisfying alternative and possibly incompatible postulates.

Among the recent nonmonotonic extensions of DLs are the formalisms for combining DLs with logic programming rules, such as for instance, [13], [30] and [25]. In [3] a non monotonic extension of DLs is proposed based on a notion of overriding, supporting normality concepts and enjoying nice computational properties. In particular, it preserves the tractability of low complexity DLs, including $\mathcal{EL}^{++}$ and $DL\text{-}lite$.

Future work may include optimizations, based on modularity as in [5], of the calculus for rational entailment, and the development of rule based inference methods for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ minimal entailment based on model checking in ASP. An upper bound on the complexity of minimal entailment for general KBs is missing. A further issue to understand is whether a materialization calculus can be defined for the preferential extensions of DLs in the $\mathcal{EL}$ family in [18, 19], whose interpretations are not required to be modular. In [19], for a preferential extension of $\mathcal{EL}^{\bot}$, with a notion on minimal model different from the one in this paper, it is shown that minimal entailment is ExpTime-hard already for simple KBs, similarly to what happens for circumscriptive KBs [4].

Apart from providing a complexity upper bound, the Datalog encoding presented in this paper is intended to provide a way to integrate the use of $\mathcal{SROEL}(\sqcap, \times)$ KBs under rational entailment with other kinds of reasoning that can be performed in ASP, and, by extending the encoding to deal with alternative models of the KB, also to allow the experimentation of alternative notions of minimal entailment, as advocated in [7].

Our approach can potentially be incorporated in systems like DReW [32], that already exploits the mapping by Krötzsch for OWL 2 EL.

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References

1. Baader, F., Brandt, S., Lutz, C.: Pushing the $\mathcal{EL}$ envelope. In: Proc IJCAI 2005. pp. 364–369 (2005)
2. Baader, F., Hollunder, B.: Priorities on defaults with prerequisites, and their application in treating specificity in terminological default logic. J. of Automated Reasoning 15(1), 41–68 (1995)
3. Bonatti, P.A., Faella, M., Petrova, I., Sauro, L.: A new semantics for overriding in description logics. Artif. Intell. 222, 1–48 (2015)
4. Bonatti, P.A., Faella, M., Sauro, L.: Defeasible inclusions in low-complexity dls. J. Artif. Intell. Res. (JAIR) 42, 719–764 (2011)
5. Bonatti, P.A., Petrova, I.M., Sauro, L.: Optimizing the computation of overriding. In: Proc. ISWC 2015. pp. 356–372 (2015)
6. Bonatti, P.A., Lutz, C., Wolter, F.: The Complexity of Circumscription in DLs. J. of Artificial Intelligence Research 35, 717–773 (2009)
7. Booth, R., Casini, G., Meyer, T., Varzinczak, I.J.: On the entailment problem for a logic of typicality. In: Proc. IJCAI 2015. pp. 2805–2811 (2015)
8. Britz, K., Heidema, J., Meyer, T.: Semantic preferential subsumption. In: Brewka, G., Lang, J. (eds.) Proc. KR 2008. pp. 476–484 (2008)
9. Casini, G., Meyer, T., Varzinczak, I.J., , Moodley, K.: Nonmonotonic Reasoning in Description Logics: Rational Closure for the ABox. In: Proc. DL 2013. pp. 600–615 (2013)
10. Casini, G., Straccia, U.: Rational Closure for Defeasible Description Logics. In: Proc. JELIA 2010. pp. 77–90. LNAI 6341, Springer (2010)
11. Donini, F.M., Nardi, D., Rosati, R.: Description logics of minimal knowledge and negation as failure. ACM Transactions on Computational Logic (ToCL) 3(2), 177–225 (2002)
12. Eiter, T., Ianni, G., Lukasiewicz, T., Schindlauer, R.: Well-founded semantics for description logic programs in the semantic web. ACM Trans. Comput. Log. 12(2), 11 (2011)
13. Eiter, T., Ianni, G., Lukasiewicz, T., Schindlauer, R., Tompits, H.: Combining answer set programming with description logics for the semantic web. Artif. Intell. 172(12-13), 1495–1539 (2008)
14. Gelfond, M.: Handbook of Knowledge Representation, chapter 7, Answer Sets. Elsevier (2007)
15. Giordano, L., Gliozzi, V.: Encoding a preferential extension of the description logic *SROIQ* into *SROIQ*. In: Proc. ISMIS 2015. LNCS, vol. 9384, pp. 248–258. Springer (2015)
16. Giordano, L., Gliozzi, V., Olivetti, N., Pozzato, G.L.: Preferential Description Logics. In: Proceedings of LPAR 2007. LNAI, vol. 4790, pp. 257–272. Springer-Verlag (2007)
17. Giordano, L., Gliozzi, V., Olivetti, N., Pozzato, G.L.: ALC+T: a preferential extension of Description Logics. Fundamenta Informaticae 96, 1–32 (2009)
18. Giordano, L., Gliozzi, V., Olivetti, N., Pozzato, G.L.: Prototypical reasoning with low complexity description logics: Preliminary results. In: Proc. LPNMR 2009. pp. 430–436 (2009)
19. Giordano, L., Gliozzi, V., Olivetti, N., Pozzato, G.L.: Reasoning about typicality in low complexity DLs: the logics $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ and *DL-Lite*$_c\mathbf{T}_{min}$. In: Walsh, T. (ed.) Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI 2011). pp. 894–899. Morgan Kaufmann, Barcelona, Spain (July 2011)

20. Giordano, L., Gliozzi, V., Olivetti, N., Pozzato, G.L.: A NonMonotonic Description Logic for Reasoning About Typicality. Artificial Intelligence 195, 165–202 (2013)
21. Giordano, L., Gliozzi, V., Olivetti, N., Pozzato, G.L.: Semantic characterization of rational closure: From propositional logic to description logics. Artif. Intell. 226, 1–33 (2015)
22. Giordano, L., Theseider Dupré, D.: Reasoning in a Rational Extension of SROEL. In: DL2016. CEUR Workshop Proceedings, vol. 1577 (2016)
23. Gottlob, G., Hernich, A., Kupke, C., Lukasiewicz, T.: Stable model semantics for guarded existential rules and description logics. In: Proc. KR 2014 (2014)
24. Ke, P., Sattler, U.: Next Steps for Description Logics of Minimal Knowledge and Negation as Failure. In: Proc. DL 2008. CEUR Workshop Proceedings, vol. 353 (2008)
25. Knorr, M., Hitzler, P., Maier, F.: Reconciling OWL and non-monotonic rules for the semantic web. In: ECAI 2012. p. 474479 (2012)
26. Kraus, S., Lehmann, D., Magidor, M.: Nonmonotonic reasoning, preferential models and cumulative logics. Artif. Intell. 44(1-2), 167–207 (1990)
27. Krötzsch, M.: Efficient inferencing for OWL EL. In: Proc. JELIA 2010. pp. 234–246 (2010)
28. Krötzsch, M.: Efficient inferencing for the description logic underlying OWL EL. Tech. Rep. 3005, Institute AIFB, Karlsruhe Institute of Technology (2010)
29. Lehmann, D., Magidor, M.: What does a conditional knowledge base entail? Artificial Intelligence 55(1), 1–60 (1992)
30. Motik, B., Rosati, R.: Reconciling Description Logics and rules. J. ACM 57(5) (2010)
31. Straccia, U.: Default inheritance reasoning in hybrid KL-ONE-style logics. In: Proc. IJCAI 1993. pp. 676–681 (1993)
32. Xiao, G., Eiter, T., Heymans, S.: The DReW system for nonmonotonic dl-programs. In: Proc. CSWS 2012. pp. 383–390 (2012)