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General composition for Symmetric Net arc functions with applications

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General composition for Symmetric Net arc functions with applications

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Abstract. Structural analysis of High-Level Petri Nets is a powerful technique, but it is less supported than in PNs. A symbolic calculus for Symmetric Nets (SNs) has been developed and implemented, which allows one to check structural properties directly on SNs without unfolding: however it is limited to a particular form of composition, restricted to functions that map to sets. To complete the calculus for more general applications the ability to solve the composition of general SN arc expressions in a symbolic way is required. In literature, a few papers show how to solve this operation for a restricted category of SN. In this report, we formalize the algebraic composition of general SN bag-functions. Some applications are also discussed.

1. Introduction

The possibility of checking some properties of PN models on their structure instead of (or before) generating their state space is a strong point in favour of the PN formalism. The extension of the structural analysis techniques to High Level Petri Nets (HLPNs), without resorting to unfolding, has been

considered in the literature and some interesting results have been published [9, 11, 13], however the applicability of the proposed methods is often limited to particular classes of HLPNs and not completely supported by software tools.

A contribution in this direction has been proposed in [6, 4] for models represented with the Symmetric Net (SN) formalism: it consists of a symbolic calculus operating on the arc expressions of the formalism, allowing to derive several interesting structural properties in a symbolic and parametric form; the software tool SNexpression[1] [7] implements the results developed so far in this direction. One limitation of such calculus concerned the composition operator, which could be applied only to the support of arc expressions (hence to functions mapping to sets): although this is sufficient for the computation of several symbolic structural relations (e.g. symbolic structural conflict or causal connection [5]) it is not enough e.g. for checking some invariant properties based on the definition of P and T-semiflows, or for applying model reduction by agglomeration of transitions, two techniques that have been applied to concurrent programs verification [11]. This report fills the gap by defining the theory allowing to symbolically apply the composition operator to general SN arc functions.

The report contains the results presented in [8], adding details about the proof of lemmas, corollaries and properties enunciated in the paper and expanding the applications section.

Report organization: Some definitions and notations are introduced in Sec. 1.1 and 1.2. In Sec. 2, 3 and 4 we describe the steps to solve the general composition of SN functions. In Sec. 5 two example applications are illustrated. Sec. 6 concludes the report and outlines directions for future work.

1.1. Basic definitions, notations, properties

A (generalized) bag b over a domain A is a map $b : A \to \mathbb{Z}$. The set of bags over A is denoted $Bag[A]$. Let $a \in A$ and $b \in Bag[A]$: we write $a \in b$ if and only if $b(a) \neq 0$. The set $\overline{b} = \{a, a \in b\}$ is the *support* of b. The bag with an empty support, called *null*, is denoted $\emptyset_A$ or just $\emptyset$ if its domain is clear from the context. Bags $b_1, b_2 \in Bag[A]$ are *disjoint* if and only if $\overline{b_1} \cap \overline{b_2} = \emptyset$. A *proper* bag is a map $b : A \to \mathbb{N}$. The set of proper bags over A is denoted $Bag^+[A]$. The *size* of $b \in Bag^+[A]$, $|b|$, is $\sum_a b(a)$. Bag b is *type-set* if $\forall a \in b\ b(a) = 1$. A bag $b \neq \emptyset$ may be represented as a formal sum $\sum_{a \in A} b(a).a$.

Let $b_1, b_2 \in Bag[A]$, $k \in \mathbb{Z}$. The scalar product $k \cdot b_1$ and the sum (diff) $b_1 \pm b_2$ are operations in $Bag[A]$ defined as $(k \cdot b_1)(a) = b_1(a) \cdot k$ and $b_1 \pm b_2(a) = b_1(a) \pm b_2(a)$, $\forall a \in A$.

Operator $+$ is associative, $-$ is associative on $Bag[A]$, but not on $Bag^+[A]$, $+$ is commutative. $\emptyset$ is the neutral element for $+,-$.

The Cartesian product is defined as follows. Let $b_1 \in Bag[A]$, $b_2 \in Bag[B]$: $b_1 \times b_2 \in Bag[A \times B]$ is $b_1 \times b_2(\langle a, b\rangle) = b_1(a) \cdot b_2(b)$, $\forall a \in A, b \in B$. The notation $\langle b_1, b_2, \ldots\rangle$ is used in place of $b_1 \times b_2 \times \ldots$. The Cartesian product is associative and may be distributed over inner $+, -$: $\langle \ldots, b_1\ op\ b_2, \ldots\rangle$, $op \in \{+, -\}$ = $\langle \ldots, b_1, \ldots\rangle\ op\ \langle \ldots, b_2, \ldots\rangle$. Finally, $\langle \ldots, k \cdot b_1, \ldots\rangle = k \cdot \langle \ldots, b_1, \ldots\rangle$.

1.2. A brief introduction to Symmetric Nets

The SN formalism belongs to the HLPN class: places and transitions are associated with a color domain $\mathcal{C}(.)$ expressing the possible *colors* of tokens and of the transition instances. Arcs are annotated by expressions, representing functions from $\mathcal{C}(t)$ to multisets $Bag^+[\mathcal{C}(p)]$; $W^-(p,t)$ and $W^+(p,t)$ denote

[1] `http://www.di.unito.it/~depierro/SNexpression/`

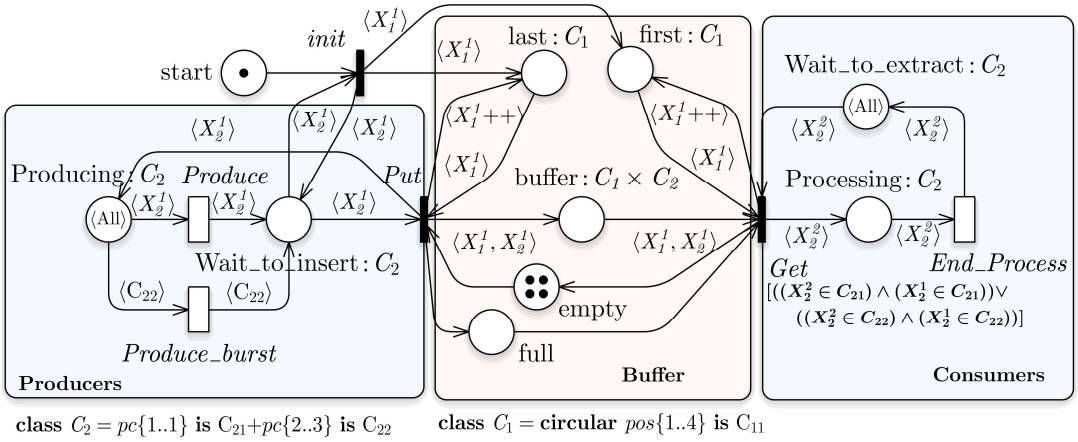

Figure 1. A SN example: Producers and Consumers

t input and output arc expressions. The structure of a color annotation is based on the definition of color classes (finite not empty sets), which may be partitioned into (static) subclasses or be circularly ordered, and on a restricted set of basic functions: $S_i, S_{i,k}, X_i, !X_i$ namely diffusion/synchronization (on class C_i or subclass $C_{i,k}$), projection X_i^j and successor $!X_i^j$, the latter only allowed for ordered classes. Color domains are defined as Cartesian products of n classes (a class may be repeated in a color domain). The arc expressions are weighted sums of tuples (Cartesian product) of basic functions (see definition 3.1). In the SN of Fig.1, representing a set of producers and a set of consumers communicating through a buffer (FIFO queue), there are two color classes, C_1 (circularly ordered, used to record the position of messages in the buffer) and C_2 (id of producers/consumers), partitioned into two subclasses C_{21}, C_{22}. The places color domains are C_2, C_1, $C_1 \times C_2$ (tokens are tuples of one or two elements); some places and one transition in this net have *neutral* color (as in PNs, e.g. place empty and transition ProduceBurst). The transitions have color domain C_2, $C_1 \times C_2$, $C_1 \times C_2^2$, depending on the projection symbols appearing on their arcs; one transition has a *guard*: a predicate restricting the allowed color instances. Arc expressions in Fig.1 are quite simple[2]: $\langle X_2^1 \rangle$, $\langle !X_1^1 \rangle$, $\langle X_1^1, X_2^2 \rangle$, $\langle S_{2,2} \rangle$. The guard is a boolean expression whose terms can be either $(X_i^j = X_i^k)$/$(!X_i^j = X_i^k)$ or $X_i^j \in C_{i,k}$. An example of transition instance is Get$(pos1, pc1, pc2)$: it satisfies the guard g (the two colors $pc2, pc3 \in C_2$ belong to the same subclass C_{22}), and the projection $X_1^1(pos1, pc2, pc3) = pos1$, while $X_1^2(pos1, pc2, pc3) = pc2$, $X_2^2(pos1, pc2, pc3) = pc3$, finally $!X_1^1(pos1, pc2, pc3) = pos2$. A more formal definition of the SN arc expressions will be given later. Produce_burst has only one instance since the functions $\langle S_{22} \rangle$ on its arcs are constant, mapping on C_{22}. The *incidence matrix* can be derived from the SN structure: in Sec.5 it will be exploited to check some invariants of this SN model.

Definition 1.1. (Incidence Matrix)
The incidence matrix $\mathbf{C}$ of a HLPN model N is a $P \times T$ matrix of functions: $\mathbf{C}[p,t] = W^+(p,t) - W^-(p,t)$. If a transition instance $t(c)$ enabled in marking $\mathbf{m}_i$ fires, the corresponding state change can be defined in terms of the incidence matrix as: $\mathbf{m}_j = \mathbf{m}_i + \mathbf{C}[.,t](c)$.

[2]In the picture the successor operator is $++$ and the diffusion/synchronization is C_{ij}.

2. Bag-expressions: general properties of composition

Operators $op \in \{+, -\}$ on functions mapping to Bags are defined as follows, in terms of Bag-operations. Let $f, h : A \to Bag[D]$, $op \in \{+, -\}$: $(f\ op\ h)(a) = f(a)\ op\ h(a)$, $(k \cdot f)(a) = k \cdot f(a), \forall a \in A$. The constant function mapping to the null bag is denoted $\epsilon_{A,D}$ (or just ϵ).

Given a family of functions $\{f_i : A \to Bag[D_i]\}$, the *function-tuple* $\langle f_1, f_2, \ldots\rangle$ is a map $A \to Bag[D_1 \times D_2 \times \ldots]$, such that $\langle f_1, f_2, \ldots\rangle(a) = \langle f_1(a), f_2(a), \ldots\rangle$. Instead of tuple notation we may use $\otimes_i f_i$.

The properties of operations on bags, as well as the type-set notion, apply to bag-functions by considering all function arguments. We call any expression built of bag-functions a bag-expression.

Definition 2.1. (function linear extension)
Let $f : A \to Bag[D]$. The linear extension $f^* : Bag[A] \to Bag[D]$ is $f^*(b) = \sum_{a \in A} b(a) \cdot f(a)$, $\forall b \in Bag[A]$.

The composition of bag-expressions builds on function linear extension.

Definition 2.2. (composition)
Let $f : A \to Bag[D]$, $h : C \to Bag[A]$. Then $f \circ h : C \to Bag[D]$ is $f \circ h(c) = f^*(h(c)), \forall c \in C$.

We shall use the same symbol for a function and its linear extension. When a function takes a bag as an argument we implicitly refer to its linear extension.

Here are some base properties of composition of bag-expressions. Their directly descend from definitions above and properties of bags. Symbols f, h, g denote *any* functions mapping to bags, with compatible arity.

Property 1. (basic properties of $\circ$)

$$(f \circ h) \circ g = f \circ (h \circ g) \quad f \circ \epsilon = \epsilon \circ h = \epsilon \qquad \lambda_1 \cdot f \circ \lambda_2 \cdot h = \lambda_1\lambda_2 \cdot f \circ h, \lambda_i \in \mathbb{Z}$$
$$(f \pm h) \circ g = f \circ g \pm h \circ g \qquad f \circ (h \pm g) = f \circ h \pm f \circ g$$

The use of generalized bags makes it possible to distribute a composition over a difference or sum during the symbolic calculus. This is very helpful because a complex composition may be reduced to an algebraic sum of simpler ones, however, for the sake of efficiency we try to minimize term expansion.

Bag-expressions may be prefixed by *filters* and suffixed by *guards*, both expressed as predicates. A $[true]$ guard/filter is usually omitted. Let $p : A \to \{true, false\}$. A filter or guard $[p]$ is a function $A \to Bag[A]$ such that $[p](a) = 1 * a$ if $p(a) = true$, $[p](a) = \emptyset$ otherwise. Let $f : A \to Bag[D]$, and p' be a predicate on D. The expressions $f[p]$ and $[p']f$ stand for $f \circ [p]$ and $[p'] \circ f$, respectively: as a consequence, the following properties of filters and guards hold.

Property 2. (basic properties of filters/guards; $=$ stands for $\equiv$)

$$[p]k \cdot f = k \cdot [p]f, k \in \mathbb{Z}$$
$$[p_1]([p_2]f) = [p_1 \wedge p_2]f \qquad (f[p_2])[p_1] = f[p_1 \wedge p_2]$$
$$\langle f[p_1], h[p_2]\rangle = \langle f, h\rangle[p_1 \wedge p_2] \qquad \langle f[p], h\rangle = \langle f, h\rangle[p]$$

The following definition characterizes an important class of (guarded) functions.

Definition 2.3. (constant-size function)
$f : A \to Bag^+[D]$ is constant-size iff $\exists n \in \mathbb{N}^+$ such that $\forall a \in A\ f(a) \neq \emptyset \Rightarrow |f(a)| = n$.

Hereafter, with f constant-size we mean $f \equiv f'[p]$, with $f'[p] = \epsilon$ iff $p = false$. The following two general properties of composition concern constant and constant-size functions.

Property 3. (composition of a constant and a constant-size function)
Let $f : B \to Bag[D]$ be a constant function so defined: $f(b) = d, \forall b \in B$. And let $h[g] : A \to Bag[B]$ be a n constant-size function, where $h[g](a) = \emptyset$ iff $g(a) = false$. Then, the composition $f \circ h[g]$ is defined as follows: $\forall a \in A$, $f \circ h[g] = n \cdot f'[g]$, where $f'[g] : A \to Bag[D]$ is such that $\forall a \in A, g(a) = true \Rightarrow f'(a) = d$.

Property 4. (composition of a tuple including a constant)
Let f' be a constant function. Then $\langle f, f' \rangle \circ h = \langle f \circ h, f' \rangle$.

Since f and f' might be tuples in turn, Property 4 applies (up to a permutation of tuple positions) to any $T \circ h$, where $T := \langle f_1, \ldots, f_m \rangle$ contains some constant components and others non-constant. We may thus reduce such a composition to $T' \circ h$, where T' is built of all and only the non-constant components f_i of T. In the sequel, we focus on this case.

3. A language for composition of SN functions

Let us anticipate the main result presented in this report by introducing the language used to calculate and express the composition of SN functions.

Definition 3.1. (Language $\mathcal{L}^c$)
Let

- $D = C_1^{e_1} \times C_2^{e_2} \times ... \times C_n^{e_n}, e_* \in \mathbb{N}$, be any color domain, i.e., a Cartesian product of colour classes (we can assume that all color domains match this ordered form; $e_i = 0$ means that C_i doesn't occur on D).
- $\mathcal{B}_i^D = \left\{ X_i^j, S_i, S_{i,k}, !^s X_i^j : D \to Bag[C_i] \right\}, i : 1, \ldots, n,\ j : 1, \ldots, e_i,\ k : 1, \ldots, ||C_i||,\ s \in \mathbb{Z}$; be the set of elementary functions, where $!^s$ denotes the s-th $mod_{|C_i|}$ successor on C_i ($!^s X_i^j \equiv !^s \circ X_i^j, !^0 = id$) [3].
- $T_j : D \to Bag[D'], T_j = \langle f_1, \ldots f_l \rangle$, and $\forall r : 1, \ldots, l$, $f_r : D \to Bag[C_i]$, $f_r = \sum_m \beta_m \cdot h_m$, $\beta_m \in \mathbb{Z}$, $h_m \in \mathcal{B}_i^D$ (where i is consistent with D'); f_r is said a class-function.
- g'_j and g_j be SN predicates on D' and D, respectively, such that all class functions appearing in $T_j[g_j]$ map to proper bags when g_j is true.

$$\mathcal{L}^c = \left\{ E : D \to Bag[D'], E = \sum_j \lambda_j [g'_j] T_j[g_j], \forall D, D' \right\},\ \lambda_j \in \mathbb{Z}$$

[3] symbols $!^s X_i^j$, with $s \neq 0$, and $S_{i,k}$ are mutually exclusive because we assume a color class is either partitioned or circularly ordered.

Any class-function f_r has as implicit guard: the guard of the tuple it belongs to.

Theorem 3.1. $\mathcal{L}^c$ is closed under composition.

The properties and lemmas presented in the sequel justify the claim above[4]. Before that, let us point out a few interesting facts about $\mathcal{L}^c$.

$\mathcal{L}^c$ includes SN arc-functions, where scalars are such that they map to $Bag^+[D']$. The possibility of prefixing function-tuples with filters makes $\mathcal{L}^c$ slightly more expressive than the language of SN arc-functions.

With respect to the language ($\mathcal{L}$) of SN structural relations defined in [4, 5] and implemented in `SNexpression` [7] the main difference is that class-functions belong to SN legacy: $S_i - X_i^j$ is a difference between elementary functions (often treated as an idiom), not a new symbol; the intersection operator (though helpful) is not part of the language; scalars are in $\mathbb{Z}$; class-functions and, consequently, function-tuples are themselves bag-expressions, whereas both in $\mathcal{L}$ and the new `GreatSPN` GUI [2] they are set-expressions. As long as we restrict to expressions mapping to proper bags, however, $\mathcal{L}$ and $\mathcal{L}^c$ are equivalent. We use $\mathcal{L}^c$ for the sake of convenience/efficiency during the symbolic calculus.

The next important properties of $\mathcal{L}^c$ directly follow from the definition of $\mathcal{L}^c$, the Cartesian product and filter/guard properties, and the basic predicate reductions listed in the Appendix A.2; the proof of all relevant properties and lemmas are in the Appendix A.1.

Property 5. (conjunctive form equivalence)
Any $E \in \mathcal{L}^c$ can be rewritten into $E' \in \mathcal{L}^c$, $E' \equiv E$, in which the predicates of filters/guards are conjunctive forms exclusively composed of (in)equality/membership clauses.

Property 6. (type-set equivalence)
Any $E \in \mathcal{L}^c$ can be rewritten into $E' \in \mathcal{L}^c$, $E' \equiv E$, in which class-functions (therefore, function-tuples) are type-set.

Property 7. (constant-size function)
Any class-function f_r is constant-size when considering its implicit guard. $|f_r|$ is the weighted algebraic sum of sizes of elementary terms of f_r.

A function-tuple not prefixed by a filter is constant-size. Its size is the product of sizes of tuple's components.

For instance, $|(S_1 - X_1^1 - X_1^2)|$, with the implicit guard $X_1^1 \neq X_1^2$, is equal to $|C_1| - 2$. This expression is also type-set.

A function-tuple prefixed by a filter may not be constant-size. An example is $[X_1^1 = X_1^2]\langle S_1 - X_1^2, S_1 - X_1^1\rangle$: the size of this function-tuple depends on whether the 1st and 2nd element of a color-tuple argument are the same or not, it is $|C| - 1$ when applied to $\langle c, c\rangle$ and $|C| - 2$ when applied to $\langle c, c'\rangle, c \neq c'$.

Property 8. (constant-size equivalence)
Any $E \in \mathcal{L}^c$ can be rewritten into $E' \in \mathcal{L}^c$, $E' \equiv E$, uniquely composed of constant-size terms.

[4] showing the closure of $\mathcal{L}^c$ under all the other operations on bag-expression (sum, difference, intersection, transpose), though much simpler, is not the focus here.

Thus, if *needed*, we can transform any expression into an equivalent sum of constant-size (and/or type-set) terms. We will return to this in Section 4.1.

When calculating a Cartesian product, we have to take care of possible filters. Since we consider domains ordered by color, we only have to deal with two cases.

Property 9. ($\mathcal{L}^c$ tuple Cartesian product)
Let $F_1, F_2 \in \mathcal{L}^c$, $F_1 : D \to D'$, $F_1 = [g_1]T_1$; $F_2 : D \to D''$, $F_2 = [g_2]T_2$, such that D', D'' are either a) disjoint or b) $D' = C_i^m$, $D'' = C_i^n$, $n, m \in \mathbb{N}^+$.

Then $\langle F_1, F_2\rangle = [g_1 \wedge {g_2}^*]\langle T_1, T_2\rangle$, where ${g_2}^* = g_2$ in case a), ${g_2}^*$ is obtained by replacing each X_i^j in g_2 with X_i^{j+m} in case b).

All results we are presenting (including those in Section 2) hold modulo a permutation of tuples.

3.1. Conventions/notations used in the sequel

Lower-case letters f, g, h, p denote class-functions and predicates, if enclosed between angular and square brackets, respectively, otherwise represent any bag-expression (like in Section 2). Upper case F denotes any expression in $\mathcal{L}^c$ whereas T any (possibly guarded) function-tuple in $\mathcal{L}^c$. $Var(f)$ denotes the set $\{X_j^i\}$ of variables (projections) appearing in f: if $f = [g]f'[g']$ then $Var(f) = Var(f') \cup Var(g')$.

Let $\overline{X}$ be a non-empty set of typed variables: $f(\overline{X})$ denotes a function such that $Var(f) = \overline{X}$. We may list all function variables, e.g., $f(X_i^j, X_h^w)$ meaning that $Var(f) = \{X_i^j, X_h^w\}$. The subset $\overline{X}_i \subseteq \overline{X}$ holds the class-C_i variables.

Index restriction ^{r} Let D: $C_1^{e_1} ... \times C_n^{e_n}$, $f = f(\overline{X}) : D \to Bag[D']$, and $e'_j = |\overline{X}_j|$ (by the way, $e'_j \leq e_j$). The index *restriction* of f is a function f^r: $C_1^{e'_1} ... \times C_n^{e'_n} \to Bag[D']$, obtained from f by replacing symbols in $X_j^i \in \overline{X}_j$, in superscript order, with $X_j^1, \ldots, X_j^{e'_j}$, for each j, $e'_j < e_j$.

For example, let $T = \langle X_1^1 + 2X_1^3, X_1^3, X_2^2\rangle$: ${C_1}^3 \times {C_2}^2 \to {C_1}^2 \times C_2$, then $T^r = \langle X_1^1 + 2X_1^2, X_1^2, X_2^1\rangle : {C_1}^2 \times C_2 \to {C_1}^2 \times C_2$.

Let $\overline{X}$ be a set $\{X_j^i\}$, $X_j^i : D \to C_j$.

Domain projection $\mathbf{\Pi}$ By convenience, we assume $\overline{X}$ ordered, first by class index then by superscript. We say the tuple $\mathbf{\Pi}_{\overline{X},D} : D \to D' = \otimes_{X_j^i \in \overline{X}} X_j^i$ projection of D (on D') induced by $\overline{X}$. D is omitted when clear from the context.

Projection image $_{\overline{X}}$ Let $T' = \langle h_1, \ldots, h_m\rangle$ with codomain D. The image of $\overline{X}$ on T' is the subtuple $T'_{\overline{X}} = \otimes_{i:\exists X_j^i \in \overline{X}} h_i$, with the same domain as T'. We denote by $T'_{\neg\overline{X}}$ the residual sub-tuple of T'.

4. Composing $\mathcal{L}^c$ expressions

Let $T : D \to Bag[D']$, $T' : D'' \to Bag[D]$, $D = C_1^{e_1} ... \times C_n^{e_n}$, $\overline{X} = Var(T) \wedge Var(T) \neq \emptyset$. Solving $T \circ T'$ means being able to rewrite this expression into $F \equiv T \circ T'$. The algorithm to solve composition operates top-down. Each step is described in detail in this section. We proceed by making assumptions more and more stringent on T, T', until we get a base form.

We initially assume to have preliminarily operated all generic reductions described in Sec.2, thus we let T be $\langle f_1, \ldots, f_n \rangle [g]$, where $\forall i \; Var(f_i) \neq \emptyset$, and $T' = \langle h_1, \ldots, h_m \rangle$. In most of following examples tuple arity is implicit.

Property 10. $T = T^r \circ \mathbf{\Pi}_{\overline{X}}$.

A first basic result says that we can solve $T \circ T'$ by considering T^r and the image of $Var(T)$ on T'.

Lemma 4.1. Let $T = T(\overline{X})$ and $T'_{\overline{X}} \neq T'$. If $T'_{\neg \overline{X}}$ is of constant-size k, then:

$$T \circ T'[g'] = k \cdot T^r \circ T'_{\overline{X}}[g']$$

Here is an example of application of Lemma 4.1:

$$\langle X_1^1, X_1^1, S_1 - X_1^3 \rangle \circ \langle S_1, 2S_1 - X_1^1, S_{1,1} + X_1^2 \rangle = (2|C_1| - 1) \cdot \langle X_1^1, X_1^1, S_1 - X_1^2 \rangle \circ \langle S_1, S_{1,1} + X_1^2 \rangle$$

Based on Lemma 4.1 (and Property 8), we focus on tuple compositions where the image of left tuple's variables coincides with the entire right tuple.

A second basic result allows one to distribute a composition over the independent parts of the left operand (up to a permutation of tuple elements).

Lemma 4.2.
Let $T = \langle F_1(\overline{X}_1), \ldots, F_h(\overline{X}_h) \rangle \wedge \forall i, j, \; i \neq j, \overline{X}_i \cap \overline{X}_j = \emptyset, \wedge T'_{\overline{X}} = T'$.

$$T \circ T' = \langle {F_1}^r \circ T'_{\overline{X}_1}, \ldots, {F_h}^r \circ T'_{\overline{X}_h} \rangle$$

In other words, one can separately compose independent sub-tuples of T with their images on T', whatever form they have. As a consequence, due to color independence, one can partition T (and T') in sub-tuples based on color-classes.

We therefore focus on function-tuples $C_i^e \rightarrow C_i^{e'}$ and hereafter omit class index in basic functions: X^j replaces X_i^j, S_j replaces $S_{i,j}$, S replaces S_i. We assume that monochromatic tuples do not include independent sub-tuples.

Lemma 4.2 exploits a nice property of linear extension of Cartesian product of bag-functions by which, if an argument is in turn a product of bags, it is possible to evaluate the product-function in a modular way.

Let us illustrate Lemma 4.2 with a non-trivial example:

$$\begin{aligned}
&\langle X^1, X^3, S - X^2, X^2 \rangle [X^1 \neq X^3 \wedge X^2 \neq X^4] \circ \langle S, 2S - X_1^1, S, S_1 + 2X_2 \rangle = \\
&\langle \langle X^1, X^3 \rangle [X^1 \neq X^3], \langle S - X^2, X^2 \rangle [X^2 \neq X^4] \rangle \circ \langle S, 2S - X^1, S, S_1 + 2X_2 \rangle \overset{Lem.4.2}{=} \\
&\langle \langle X^1, X^2 \rangle [X^1 \neq X^2] \circ \langle S, S \rangle, \langle S - X^1, X^1 \rangle [X^1 \neq X^2] \circ \langle 2S - X^1, S_1 + 2X_2 \rangle \rangle
\end{aligned}$$

The two compositions that we ended up with are representatives of the base cases of tuple composition we have to solve.

4.1. Tuple composition's base cases

We can identify a few base cases of function-tuple composition as a result of Lemma 4.2 application. One in which the left-hand tuple is a one-variable function and the right tuple is a singleton. The others in which there is an infix predicate (guard/filter) that cannot be eliminated. The presence of an infix predicate complicates the solution of a composition. Based on properties of filters and filter reduction rules we may reduce such expressions to a particular form. The following statements assume that the right operand of a composition (T') is possibly followed by a guard.

Definition 4.1. (tuple prefix/composition simple form(s))
Let $T' = \langle h_1, \ldots, h_m \rangle$. A tuple-prefix $[g]T'$ is simple if and only if

1. g is a conjunction of (in)equalities, such that for each clause $(X^i \stackrel{\neq}{=}!^s X^j)$ in g: h_i is type-set, $|h_i| > 1$, and $h_i =!^s h_j$
2. let $g_= \cup g_\neq$ be the partition of g in equalities and inequalities
   (a) if $g_\neq \neq \emptyset$, each partition $g_1 \cup g_2$ of $g_\neq$ is such that $Var(g_1) \cap Var(g_2) \neq \emptyset$
   (b) if $g_= \neq \emptyset$, then $g_=$ may be partitioned in $g^i_=, \ldots, g^w_=$, $w \in \mathbb{N}^+$, such that $\forall i, j,\ i \neq j$, $Var(g^i_=) \cap Var(g^j_=) = \emptyset$
   $\forall g^i_=\ Var(g^i_=) \cap Var(g_\neq) = \{X^j\}$

$T[g] \circ T'$ is a simple form of composition if and only if $[g] \circ T$' is a simple tuple-prefix and $\forall i\ |Var(g^i_=) \cap Var(T)| \leq 1$.

Claim 1. We can rewrite any composition $T[g] \circ T'$ that cannot be decomposed according to Lemma 4.2 in terms matching Definition 4.1 (see rewriting rules in the Appendix A.2).

In other words, if required we may assume that infix predicates are made of (in)equalities between projections which point to equal (modulo-successor) type-set class-functions of T' of size > 1. Equalities define equivalence classes of projections, one representative of each class must appear in inequalities, and at most one occur on T. For example: Let ($|C_2| > 1$):

$$\langle S_2 - X^3, X^1 \rangle [X^1 \neq X^2 \wedge X^1 \in C_2 \wedge X^1 = X^3] \circ \langle S, S, S + X^1 \rangle \rightarrow$$
$$\langle S_2 - X^1, X^1 \rangle [X^1 \neq X^2 \wedge X^1 = X^3] \circ \langle S_2, S_2, S_2 \rangle +$$
$$\langle S_2 - X^1, X^1 \rangle [X^1 = X^3] \circ \langle S_2, S - S_2, S_2 \rangle + \langle S_2 - X^1, X^1 \rangle \circ \langle X^1, S - X^1, X^1 \rangle [X^1 \in C_2]$$

Tuples prefixed by a filter matching Definition 4.1 (part 1) have an important property.

Property 11. A simple tuple-prefix $[p]T : D \rightarrow Bag[D']$ is constant-size.

Four base cases have to be considered, they are listed below.

1. $T(X^1) \circ \langle h \rangle$
2. $T(\overline{X})[g] \circ T'$, with $|\overline{X}| > 1$ and $\overline{X} \supseteq Var(g)$
3. $T(\overline{X})[g] \circ T'$, with $\overline{X} \subset Var(g)$
4. $T[g] \circ T'$, with $Var(T) \cap Var(g) = \emptyset$

4.1.1. Case 1: $T(X^1) \circ \langle h \rangle$

In absence of infix predicates, we can always reduce to one such form due to the distribution property of sum/diff. For example:

$$\langle S - X^1 + 2X^2, X^2 \rangle \circ \langle h_1, h_2 \rangle = \langle (S - X^1) \circ \langle h_1 \rangle, X^1 \circ \langle h_2 \rangle \rangle + 2|h_1| \cdot \langle X^1, X^1 \rangle \circ \langle h_2 \rangle$$

Case 1.a: $|h| = 1$; this is the simplest situation

Property 12. Let $|h| = 1$. Then $F(X^1) \circ \langle h \rangle$ is obtained by replacing each occurrence of symbol X^1 in F with h.

It directly follows from the definition of composition.

For example, $\langle S + X^1, X^1 \rangle \circ \langle h_2 \rangle$, $|h_2| = 1 \longrightarrow \langle S + h_2, h_2 \rangle$.

Case 1.b: $|h| > 1$. The basic compositions are summarized in the following table (omitting tuple notation):

Basic composition rules and some useful identities			
$X^1 \circ h = h$	$!^s X^1 \circ h = !^s h$	$S - X^1 \circ h = \|h\| \cdot S - h$	$S - !^r X^1 \circ h = \|h\| \cdot S - !^r h$
$!^r S = S$	$!^s !^r X^i = !^{s+r} X^i$	$!^r (h_1 \pm h_2) = !^r h_1 \pm !^r h_2$	

If $T = \langle f_1(X^1) \rangle$ the above basic rules (and property 4) are enough. For example, letting $|C_1| = 3$: $\langle 2S - X^1 - !X^1 \rangle \circ \langle S - X^2 \rangle \rightarrow \langle 4S - S + X^2 - S + !X^2 \rangle \equiv \langle 2S + X^2 + !X^2 \rangle$.

Repetition of a projection in T Let $T = \langle f_1(X^1), \ldots, f_m(X^1) \rangle$, where (without loss of generality due to Property 6) $f_i(X^1) = !^{s_i} X^1$, $\forall i$ (symbol $!^s X^i$ from now on denotes a projection *possibly* prefixed by the s-th successor). We consider first an unordered color class, then we generalize.

Property 13. Let h be type-set. $\underbrace{\langle X^1, \ldots, X^1 \rangle}_{m>1} \circ \langle h \rangle = [\bigwedge_{i:2\ldots m} X^1 = X^i] \underbrace{\langle h, \ldots, h \rangle}_{m}$

Here are some non-elementary, type-set class-functions[5]:

$S - X^1$; $S - X^1 - X^2[X^1 \neq X^2]$; $S_i - X^1[X^1 \in C_i]$; $S - X^1 - !X^1$.

We can extend the above outcome to any color class.

Property 14. Let h be type-set.

$$\langle !^{r_1} X^1, \ldots, !^{r_m} X^1 \rangle \circ \langle h \rangle = [\bigwedge_{i:2\ldots m} X^1 = !^{(r_1 - r_i)} X^i] \langle !^{r_1} h, \ldots, !^{r_m} h \rangle$$

As an example:

$\langle !X^1, X^1, !X^1 \rangle \circ \langle S - X^2 \rangle = [X^1 = !X^2 \wedge X^1 = X^3] \langle S - !X^2, S - X^2, S - !X^2 \rangle$.

In some cases *may* treat the idiom $S - X^1$ as a single function for convenience. Here are two situations that are more efficiently processed applying the following properties, although they could be solved applying the previous rules. Without loss of generality, in both cases $S - X^1$ is the last element of T.

[5] if b is any bag, $\langle X^1, \ldots, X^1 \rangle(b)$ is obtained from $[\bigwedge_{i:2\ldots m} X^1 = X^i] \langle b, \ldots, b \rangle$ by applying the m^{th} root to multiplicities of bag elements

Property 15. Let h be a type-set function.
$\underbrace{\langle X^1, \ldots, X^1, S - X^1\rangle}_{m} \circ \langle h\rangle = [X^1 \neq X^m \bigwedge_{i:2\ldots m-1} X^1 = X^i] \underbrace{\langle h, \ldots, h, S\rangle}_{m}$

If there are no repetitions of X^1 then h may be any function.

Property 16. $\langle X^1, S - X^1\rangle \circ \langle h\rangle = [X^1 \neq X^2]\langle h, S\rangle$

We may extend the two properties above to any class by expressing (in-)equalities like in Property 14. As an example:

$\langle S - !X^1, X^1\rangle \circ (S + X^2) = [X^1 \neq !X^2]\langle S, S + X^2\rangle$.

Finally, we can always reduce complex compositions to simpler (solvable) forms by distributing the composition over a sum/difference. For example:

$\langle S - !X^1, X^1, S - X^1\rangle \circ \langle h\rangle = \langle S, X^1, S - X^1\rangle \circ \langle h\rangle - \langle !X^1, X^1, S - X^1\rangle \circ \langle h\rangle$.

4.1.2. Case 2: $T(\overline{X})[g] \circ T'$, $|\overline{X}| > 1$, $\overline{X} \supseteq Var(g)$

In this and in the next case we may assume, without loss of generality, that g is composed of (in-)equalities and $T = \langle f_1, \ldots, f_m\rangle$, where $f_i = !^{s_i}X^j$, $\forall i$. Consider, for example:

$$\langle X^2, !X^2, X^3, !^2X^1\rangle[X^1 \neq X^3 \wedge X^1 \neq !X^2] \circ \langle h_1, h_2, h_3\rangle$$

This expression doesn't match $\mathcal{L}^c$. However, we can transform the guard in between into a filter prefixing the left tuple through an index substitution, i.e., an injective map $\phi : Var(T) \to \{1 \ldots m\}$ associating each X^i to the position of any of its occurrences in T. The picture below illustrates the idea.

$$\langle X^2, !X^2, X^3, !^2X^1\rangle[X^1 \neq X^3 \wedge X^1 \neq !X^2].$$

The following rule formalizes the move of a clause of a guard g to a filter prefixing T ($X^i \stackrel{\neq}{=} !^kX^j \equiv X^j \stackrel{\neq}{=} !^{-k}X^i$).

Lemma 4.3. (transforming the infix predicate into a filter)
Let $\phi(i)$, $\phi(j)$ be any two occurrences of variables X^i, X^j, $i \neq j$, in tuple T.

$$\underbrace{\langle \ldots, \overset{\phi(i)}{!^rX^i}, \ldots, \overset{\phi(j)}{!^sX^j}, \ldots\rangle}_{T}[X^i \stackrel{\neq}{=} !^kX^j, \ldots] \quad \longrightarrow \quad [X^{\phi(i)} \stackrel{\neq}{=} !^{k+r-s}X^{\phi(j)}]T[\ldots]$$

Corollary 4.1. Let $T[g]$, with $T = \langle f_1, \ldots, f_m\rangle$ and g be a set of (in)equalities. If $\forall X^i \in Var(g)$ there is f_j, $f_j = !^sX^i$, then $\exists p\, T[g] \equiv [p]T$.

Therefore, if $Var(T) \supseteq Var(g)$ the reiterated application of Lemma 4.3 eventually removes the infix guard from $T(\overline{X})[g] \circ T'$.

Applying Lemma 4.3 to the last example,we can proceed with Lemma 4.2.

$$\xrightarrow{Lem.4.3} \quad [X^4 \neq !^2X^3 \wedge X^4 \neq !^2X^2]\langle X^2, !X^2, X^3, !^2X^1\rangle \circ \langle h_1, h_2, h_3\rangle$$

$$\xrightarrow{Lem.4.2} \quad [X^4 \neq !^2X^3 \wedge X^4 \neq !^2X^2]\langle\langle X^1, !X^1\rangle \circ \langle h_2\rangle, X^1 \circ \langle h_3\rangle, !^2X^1 \circ \langle h_1\rangle\rangle$$

4.1.3. Case 3: $T(\overline{X})[g] \circ T'$, $\overline{X} \subset Var(g)$

This situation is the most complex one. The reason is that we must project on $\overline{X} = Var(T)$ the application of the filter g on T', in a symbolic way.

We assume that g is no further reducible, the composition to solve matches Definition 4.1 and none of lemmas presented so far applies (in particular Lemma 3.1, thus $T'_{Var(g)} = T'$).

Let $A \subseteq Var(g)$: we say $g_A = \{(X^i \stackrel{\neq}{=} !^r X^j) \in g | X^i, X^j \in A\}$ the restriction of g to variables A.

A first simplification comes from the fact that we may take out equalities of g. We recall that f^r denotes the index-restriction of f.

Property 17. Let $[g]T'$ meet Definition 4.1, $g_= \neq \emptyset$, $\overline{X}' = \overline{X} \cup Var(g_\neq)$. Then

$$T(\overline{X})[g] \circ T' = (T[g_\neq])^r \circ T'_{\overline{X}'}$$

This nice property stems from the fact that (by Definition 4.1) T' components are equal (modulo successor) and in T only one symbol per equivalence class is used. Here are some examples of application of Property 17. For simplicity, in all the following examples $\overline{X} = \{X^1, \ldots, X^m\}$ ($T = T^r$).

1) $\langle X^1, S - X^1\rangle[X^1 = X^2, X^1 = !X^3] \circ \langle S - !X^2, S - !X^2, S - X^2\rangle$ $\xrightarrow{Prop.17}$
   $\langle X^1, S - X^1\rangle \circ \langle S - !X^2\rangle \xrightarrow{Prop.16} [X^1 \neq X^2]\langle S - !X^2, S\rangle$

2) $\langle X^1, !X^2\rangle[X^1 \neq !X^2, X^2 = X^3] \circ \langle S - !X^2, S - X^2, S - X^2\rangle$ $\xrightarrow{Prop.17}$
   $\langle X^1, !X^2\rangle[X^1 \neq !X^2] \circ \langle S - !X^2, S - X^2\rangle \xrightarrow{Lem.4.3, Lem.4.2} [X^1 \neq X^2]\langle S - !X^2, S - !X^2\rangle$

3) $\langle X^2, X^2\rangle[X^1 \neq X^2, X^2 \neq X^4, X^1 = X^3] \circ \langle S_1, S_1, S_1, S_1\rangle$ $\xrightarrow{Prop.17}$
   $(\langle X^2, X^2\rangle[X^1 \neq X^2, X^2 \neq X^4])^r \circ \langle S_1, S_1, S_1, S_1\rangle_{\{X^1, X^2, X^4\}}$ $\equiv$
   $\langle X^2, X^2\rangle[X^1 \neq X^2, X^2 \neq X^3] \circ \langle S_1, S_1, S_1\rangle$ (non-reducible with the available rules)

Main sub-case: $g = g_\neq$ We may therefore assume that g is a non-empty set of inequalities and focus on $\mathbf{\Pi}_{\overline{X}} \circ [g]T'$.

In general, we figure out that it holds

$$\overline{\mathbf{\Pi}_{\overline{X}} \circ [g]T'} \subseteq \overline{[g_{\overline{X}}]^r\ T'_{\overline{X}}} \tag{1}$$

Equation 1 says that, disregarding bag multiplicity, the projection of $[g]T'$ on $Var(T)$ is included in the restriction of $[g]$ to $Var(T)$ which applies to the image of $Var(T)$ on T'. It follows from our assumptions, by which: $g \Rightarrow g_{\overline{X}}$.

In the sequel, we characterize particular forms $[g]T'$ verifying

$$\mathbf{\Pi}_{\overline{X}} \circ [g]T' = k \cdot [g_{\overline{X}}]^r \; T'_{\overline{X}},\; k \in \mathbb{N} \tag{2}$$

showing that (modulo some rewriting) we can always reduce our expression to such a form. An immediate corollary of (2) is:

Corollary 4.2. Let $T = T(\overline{X})$, $\overline{X} \subset Var(g)$.

If (2) holds and $[g] \circ T'$ is a simple tuple-prefix (Definition 4.1) then:

$T(\overline{X})[g] \circ T' = k \cdot T^r[g_{\overline{X}}]^r \circ T'_{\overline{X}}$, where $k = \frac{|[g]T'|}{|[g_{\overline{X}}]^r T'_{\overline{X}}|}$ if $|[g_{\overline{X}}]^r T'_{\overline{X}}| \neq 0$, otherwise $k = 0$

Corollary 4.2 outlines that we bring a composition to a form where the variables of the left tuple are a super-set of those of the infix predicate.

Consider this simple but interesting case where $T[g] : C^3 \to C^2$, $T' : C \to C^3$.

$$\underset{T}{\langle X^1, X^2 \rangle}[X^1 \neq X^3 \underset{g}{\wedge} X^2 \neq X^3] \circ \langle S - X^1, S \underset{T'}{-} X^1 S - X^1 \rangle \qquad |C| > 2$$

In this case condition (2) is not verified: $\mathbf{\Pi}_{\{X^1, X^2\}} \circ [g]T'(c)$ turns out to be, for any $c \in C$, $|C| > 2$:

$$(|C| - 2) * \sum_{c' \in C, c' \neq c} \langle c', c' \rangle + (|C| - 3) * \sum_{c', c'' \in C, c' \neq c, c'' \neq c, c' \neq c''} \langle c', c'' \rangle$$

The restriction of $g[T']$ to $\overline{X} = \{X^1, X^2\}$ used on the right-hand side of (2), instead, is $\langle S - X^1, S - X^1 \rangle$ (in this particular case, $g_{\overline{X}} = \{\}$). As the example suggests, we should distinguish (in g) the case $X^1 = X^2$ from the case $X^1 \neq X^2$.

Since we are assuming that the requirements of Definition 4.1 are met, we may conveniently study $[g]T'$ as a system of inequalities (in case of ordered classes, the successors are all expressed modulo-$|C|$) among $Var(g)$ variables, with the implicit constraints $X^i \in h_i(c)$, $\forall X^i \in Var(g)$, h_i being the i-th component of T'. Thus, $[g]T'(c)$ is a type-set bag which corresponds to the system's solutions, while $\mathbf{\Pi}_{\overline{X}} \circ [g]T'(c)$ is a bag representing their projection on subset $\overline{X}$.

Due to the symmetry of g and to the fact that functions h_i are equal (modulo successor), we can abstract from both c and i and consider any h_i as a (parametric) set of known size.

We put some conditions on g to ensure that the projection of the inequality system's solutions on $\overline{X}$ is the parametric multi-set consisting of k instances of the solutions of the *sub-system* restricted to $\overline{X}$, i.e., $[g_{\overline{X}}]^r \; T'_{\overline{X}}$.

We consider first an unordered color-class C. In that case, $h_i = h$, $\forall i$, and we may represent g as an undirected simple graph whose vertices are $Var(g)$ and whose edges connect vertices corresponding to variables that are required to be different by a term of g. Abusing notation, we denote the same way g and the corresponding graph, leaving the context to disambiguate. Thus, we may interpret $g_{\overline{X}}$ as a (proper) subgraph of g.

A few basic results of graph-colouring theory turn out to be useful for our purposes (refer to [10] for all the details). Given a graph $G = (V_G, E_G)$, where V_G is a non-empty finite set and E_G a set of unordered pairs xy, $x, y \in V(G)$, $x \neq y$, and $\lambda \in \mathbb{N}$, we define a λ-colouring of G as a map $\varphi : V_G \to \{1, 2, \ldots, \lambda\}$ that assigns adjacent vertices of G different values. We are interested in the number of different λ-colourings of G, denoted $P(G, \lambda)$. The minimum value of λ such that G admits

a λ-colouring is the *chromatic number of* G, denoted χ_G. The value $|V(G)|$ is the order of G. We say that G is empty if $E(G) = \emptyset$, complete (or clique) if any two vertices are adjacent. A clique of order n is denoted K_n. We use a few operations on graphs: let x, y be two vertices of G, $G \cdot xy$, $xy \notin E(G)$, is the graph obtained from G by merging x and y and leaving one occurrence of possible resulting multiple edges; $G - xy$, $xy \in E(G)$, the graph obtained by removing edge xy; $G + xy$, $xy \notin E(G)$, the graph obtained by adding edge xy.

$P(G, \lambda)$ can be expressed as a polynomial in λ, called chromatic polynomial of G. This is interesting, since it gives us the possibility to express a composition's result in a parametric way. Computing χ_G is around $O(2^n)$, and computing $P(G, \lambda)$ is at least as complex. [10] shows, however, that for large classes of graphs you can compute $P(G, \lambda)$ very efficiently, e.g., exploiting their modular structure. How to compute the chromatic polynomial is out of the scope of the report, however, here are a few intuitive properties used in the sequel.

- (Fundamental reduction theorem) let x, y be two non-adjacent vertices of G: $P(G, \lambda) = P(G \cdot xy, \lambda) + P(G + xy, \lambda)$.
- Let G include K_r and G' be obtained from G by adding a new vertex x which is (uniquely) linked to all vertices of K_r. Then $P(G', \lambda) = P(G, \lambda)(\lambda - r)$
- Some known polynomials: $P(K_r, \lambda) = \lambda(\lambda - 1) \cdots (\lambda - r + 1)$; if G is an empty graph of order n, $P(G, \lambda) = \lambda^n$; ...

Any solution of the inequality system $[g]T'$ (g, from now on) corresponds in fact to a λ-colouring of g with $\lambda = |h|$, therefore, denoting with $Sol(g, \lambda)$ the number of solutions of the inequality system, $P(g, \lambda) = Sol(g, \lambda)$.

Lemma 4.4. Let $[g]T'$ be a simple tuple-prefix, $\overline{X} \subset Var(g)$, and $g_{\overline{X}}$ be a clique. Then, equation (2) holds and (Corollary 4.2) $k = \frac{P(g,\lambda)}{P(K_{|\overline{X}|},\lambda)}$ if $\lambda \geq |\overline{X}|$, otherwise $k = 0$, where $\lambda = |h_i|$, for any h_i in T'.

It is sufficient to observe that for any two distinct λ-colourings of $g_{\overline{X}}$ there are (obviously) the same number of λ-colourings of g that include them. Moreover, we know that $\chi_{K_r} = r$ and $\chi_g \geq \chi_{g_{\overline{X}}}$.

We are always able to reduce a composition to the condition that meets Lemma 4.4 by linking non-adjacent vertices of $g_{\overline{X}}$. Two such vertices are two unrelated variables $X^i, X^j \in \overline{X}$, therefore we rewrite $T \circ [g][T']$ into the equivalent sum $T \circ [g \wedge X^i \neq X^j]T' + T \circ [g \wedge X^i = X^j]T'$. Note that $g \wedge X^i = X^j$, after symbol replacement and removal of redundant inequalities (according to Definition 4.1) exactly corresponds to $g \cdot X^i\ X^j$.

By recursively rewriting and applying variable substitutions accordingly in T we eventually get sub-compositions solvable with the lemmas above. Let us instantiate this simple procedure and Lemma 4.4 on the last example.

$$
\begin{aligned}
&\underset{T}{\langle X^1, X^2\rangle}[X^1 \neq X^3 \wedge \underset{g}{X^2 \neq X^3}] \circ \langle S - X^1, \underset{T'}{S - X^1 S}, -X^1\rangle \ \ \lambda = |C| - 1\ (> 1)\\
&\equiv \langle X^1, X^2\rangle[X^1 \neq X^3 \wedge X^2 \neq X^3 \wedge X^1 \neq X^2] \circ T' + \langle X^1, X^1\rangle[X^1 \neq X^3] \circ T'\\
&\equiv \langle X^1, X^2\rangle \circ (\lambda - 2)[X^1 \neq X^2]\langle S - X^1, S - X^1\rangle + \langle X^1, X^1\rangle \circ (\lambda - 1)\langle S - X^1\rangle\\
&\equiv (\lambda - 2)[X^1 \neq X^2]\langle S - X^1, S - X^1\rangle + (\lambda - 1)[X^1 = X^2]\langle S - X^1, S - X^1\rangle
\end{aligned}
$$

where $\lambda - 2 = \frac{P(K_3,\lambda)}{P(K_2,\lambda)}$ and $\lambda - 1 = \frac{P(K_2,\lambda)}{P(K_1,\lambda)}$.

Dealing with an ordered class C. What if predicate g of the filter prefixing tuple T' is defined on an ordered class? In theory, the situation is even simpler because we can always rewrite any inequality into a disjunction of equalities: for example, if $|C| = 3$: $X^1 \neq !X^2 \equiv X^1 =!^2 X^2 \vee X^1 = X^2$. So, by expanding $T \circ [g]T'$ accordingly we eventually get a solvable form.

This approach, however, may be inefficient and is not parametric. Therefore, we slightly extend the technique based on graph-representation of g to ordered classes. The graph representing g now has as vertices the symbols $Symb(g) = \{!^r X^j\}$ occurring in g and as edges, in addition to inequalities in g, those implied by them: for any two symbols $!^r X^j, !^s X^j \in Symb(g)$, there is a corresponding edge in the graph[6]. In this case, a λ-colouring of g doesn't necessarily match a solution of the inequality system, i.e., $P(g, \lambda) \geq Sol(g, \lambda)$,

Due to the circularity of C there are a number of equivalent representations for g. A first result concerns those cases where g may be expressed in a form such that $|Var(g)| = |Symb(g)|$, i.e., for each variable there is one corresponding symbol. In this case the previous results, in particular Lemma 4.4, still hold. When $|Var(g)| < |Symb(g)|$ instead, an extension of Lemma 4.4 is needed.

Lemma 4.5. Let $[g]T'$ be a simple tuple-prefix and $\overline{X} \subset Var(g)$. If a) for each $X^i \in Var(g) \backslash \overline{X}$ any two inequalities between $\overline{X}$ and X^i use the same symbol $!^r X^i$ and b) $g_{\overline{X}}$ is a clique, then equation (2) holds and $k = \frac{Sol(g,\lambda)}{Sol(g_{\overline{X}},\lambda)}$ if $Sol(g_{\overline{X}}, \lambda) > 0$, otherwise $k = 0$, where $\lambda = |h_i|$, for any h_i in T'.

We can always rewrite any predicate g so that it meets condition a) of Lemma 4.5. Condition a) is redundant if $|\overline{X}| = 1$.

It is worthwhile pointing out that for the computation of $Sol(g, \lambda)$, for any inequality graph g, we can use similar basic techniques as for the chromatic polynomial.

A few examples of applications of the last two lemmas are in Appendix A.4.

4.1.4. Case 4: $T[g] \circ T',\ Var(T) \cap Var(g) = \emptyset$

. We may reduce this case to one solvable with Lemma 4.1. Indeed, we can express (modulo a tuple permutation) $[g] \circ T'$ as a Cartesian product $\langle [g]^r T'_{Var(g)}, T'_{\neg Var(g)} \rangle$, where $T'_{Var(g)}$ is the image of g on T'. If $[g]^r T'_{Var(g)}$ is simple (therefore, constant-size) we can directly use Lemma 4.1. For instance ($\lambda = |C|$):

$$\langle X^1 \rangle [X^2 \neq X^3] \circ \langle S - X^1, S, S \rangle \rightarrow \langle X^1 \rangle \circ \langle S - X^1, [X^1 \neq X^2] \langle S, S \rangle \rangle \quad \rightarrow$$
$$\lambda(\lambda - 1) \cdot \langle S - X^1 \rangle$$

An example outlining the complete composition procedure is presented in the Appendix A.4.

5. Applications

This section presents two application examples: the symbolic verification of invariants in SN models, and the reduction of nets by agglomeration of transitions, enabling more efficient qualitative analysis

[6]Let $succ_{min}$ and $succ_{max}$ be the smallest and largest successor index in the formula; we assume that $succ_{max} - succ_{min} < |C|$, so that for each two symbols $!^r X^j, !^s X^i$, and for each c, $!^r X^j(c) \neq !^s X^j(c)$; as a consequence we may have consider different cases, depending on the size of C

(as in behavior preserving reductions [3, 11]) and quantitative analysis (e.g. elimination of immediate transitions in Stochastic SN models, extending the technique defined for GSPNs in [1]).

Verification of P and T-invariants P and T-semiflows inducing invariant properties in PNs have been introduced in mid 80s and extended to HLPN in 90s [13]. Algorithms to compute a generating family of P and T-semiflows of PN exist [9] and are implemented in several tools; they can be applied to the unfolding of an HLPN model, while the automatic derivation of high-level (symbolic) P or T-semiflows is still an open problem, unless restrictions are imposed on the HLPN formalism (as in [12]). Often the modeler is aware of which invariant properties should satisfy a given model, thus the ability to automatically check whether a symbolically defined P or T-indexed vector of functions corresponds to a P or T-semiflow is interesting. The symbolic calculus implemented in the SNexpression tool [7], extended with the composition presented in this report, allows to implement such automatic check. Let us illustrate how this can be done on the example net in Fig. 1 (for the sake of space only T-invariants will be shown).

Definition 5.1. (P/T-semiflow of HLPN models)
Let $\mathbf{x}$ be a P-indexed vector of functions with color domain $\mathcal{C}(\mathbf{x})$, $\mathbf{x}[p] : \mathcal{C}(p) \rightarrow Bag(\mathcal{C}(\mathbf{x})), p \in P$; $\mathbf{x}$ is a P-semiflow if $\mathbf{x} \circ \mathbf{C} = \mathbf{null_x}$. A P-semiflow induces a marking invariant since $\mathbf{x} \circ \mathbf{m} = \mathbf{x} \circ \mathbf{m}_0, \forall \mathbf{m}$ reachable from $\mathbf{m}_0$.

Let $\mathbf{y}$ be a T-indexed vector of functions with color domain $\mathcal{C}(\mathbf{y})$, $\mathbf{y}[t] : \mathcal{C}(\mathbf{y}) \rightarrow Bag[\mathcal{C}([t)], t \in T$; $\mathbf{y}$ is a T-semiflow if $\mathbf{C} \circ \mathbf{y} = \mathbf{null_y}$. A T-semiflow defines a (parametric) set of transition instances; given a color $c \in \mathcal{C}(\mathbf{y})$ if all the transition instances in $\mathbf{y}(c)$ can fire (in any order) starting from marking $\mathbf{m}$, they bring back the model to the same marking $\mathbf{m}$.

Where $\mathbf{null_x}$ is a T-indexed vector of functions such that $\mathbf{null_x}[t]$ is a constant function mapping any $c \in \mathcal{C}(t)$ into a constant empty Bag on $\mathcal{C}(\mathbf{x})$. While $\mathbf{null_y}$ is a P-indexed vector of functions such that $\mathbf{null_y}[p]$ is a constant function mapping any $c \in \mathcal{C}(\mathbf{y})$ into a constant empty Bag on $\mathcal{C}(p)$.

In Table 1 a few P-semiflows and two T-semiflows of the producers-consumers SN are listed. P1 means that the set of producers remains constant, equal to C_2, P3 has a similar meaning, but concerns the Consumers. P2a and P2b are related, in fact they both state that the sum of empty and full positions in the buffer is constant (equal to $|C_1|$). Observe that the dummy color class $C_\bullet$ of cardinality one is used to indicate plain, black tokens. Finally P4a and P4b indicate that tokens in place start and in places first and last are related: given the initial marking, either there is a single token in start or a single token in both first and last.

Let us consider the two T-invariants. Their interpretation is quite simple[7]: in order to reproduce a marking the pointers first and last should be increased by one $|C1|$ times, hence the producers should produce $|C1|$ items and put them into the buffer, and the consumers should get $|C1|$ items from the buffer. In T1 the same producer fills completely the buffer and a consumer with the same id of the producer empties it (it represents $|C2|$ invariants in the unfolded net). In T2 any combination of $|C1|$ producers and $|C1|$ consumers (satisfying the invariant guard g') is possible.

Let us write down the expressions that allow to verify that those vectors correspond to parametric P-semiflows: $\forall t\ in T : \sum_{p \in P} P_i \circ \mathbf{C}[p, t] = null_{\mathcal{C}(P_i)}$. Observe that the function $P_i[p]$ has domain $\mathcal{C}(p)$

[7]Initialization transition init cannot belong to any T-invariant, while additional invariants involving ProduceBurst exist, but are not included for the sake of space.

Place	P1	P2a	P2b
	C_2	$C_\bullet$ ($\lvert C_\bullet\rvert = 1$)	
Producing (C_2)	$\langle X_2^1\rangle$		
Wait_to_insert (C_2)	$\langle X_2^1\rangle$		
buffer (C_1, C_2)		$\langle S\_C_\bullet\rangle$	
empty ($C_\bullet$)		$\langle S\_C_\bullet\rangle$	$\langle S\_C_\bullet\rangle$
full ($C_\bullet$)			$\langle S\_C_\bullet\rangle$
Place	P3	P4a	P4b
	C_2	$C_\bullet$ ($\lvert C_\bullet\rvert = 1$)	
start ($C_\bullet$)		$\langle S\_C_\bullet\rangle$	$\langle S\_C_\bullet\rangle$
first (C_1)		$\langle S\_C_\bullet\rangle$	
last (C_1)			$\langle S\_C_\bullet\rangle$
Wait_to_extract (C_2)	$\langle X_2^1\rangle$		
Processing (C_2)	$\langle X_2^1\rangle$		

Transition (Domain)	T1 C_2	T2 (Hp: $\lvert C1\rvert = 4$) $C_2^8, C_1\ [g']$
Produce (C_2)	$\lvert C_1\rvert\langle X_2^1\rangle$	$\langle X_2^1\rangle + \langle X_2^3\rangle + \langle X_2^5\rangle + \langle X_2^7\rangle$
Put (C_1, C_2)	$\langle S_{C_1}, X_2^1\rangle$	$\sum_{i=0}^{3}\langle !i\ X_1^1, X_2^{2i+1}\rangle$
Get $[g](C_1, C_2^2)$	$\langle S_{C_1}, X_2^1, X_2^1\rangle$	$\sum_{i=0}^{3}\langle !i\ X_1^1, X_2^{2i+1}, X_2^{2(i+1)}\rangle$
End_Process (C_2)	$\lvert C_1\rvert\langle X_2^1\rangle$	$\langle X_2^2\rangle + \langle X_2^4\rangle + \langle X_2^6\rangle + \langle X_2^8\rangle$

$$g : (X_2^1 \in C_{21} \wedge X_2^2 \in C_{21}) \vee (X_2^1 \in C_{22} \wedge X_2^2 \in C_{22})$$
$$p(k) = \bigwedge_{i=0}^{k}\left(\bigvee_{j=1}^{2} X_2^{2i+1} \in C_{2j} \wedge X_2^{2(i+1)} \in C_{2j}\right);\ g = p(0);\ g' = p(|C_1| - 1)$$

Table 1. P and T-semiflows of the Producers-Consumers SN

and co-domain $\mathcal{C}(P_i)$ (so that the terms in the summation are homogeneous). Instead The expressions allowing to verify that vectors T_j correspond to parametric T-semiflows are: $\forall p \in P, \sum_{t\in T}\mathbf{C}[p,t] \circ T_j[t] = \epsilon_{\mathcal{C}(p)}$; the function $T_j[t]$ has domain $\mathcal{C}(T_i)$ and codomain $\mathcal{C}(t)$, so that the domains of the functions to be composed are coherent. Observe that T_2 represents 625 T-invariants in the unfolded net when $|C_1| = 4$, $|C_{21}| = 1$ and $|C_{22}| = 2$.

Let us consider the formulae that allow us to check P2a. P2a[buffer]$(X_1^1, X_2^1) = \langle S_{C_\bullet}\rangle$.
P2a[empty]$(X_\epsilon) = \langle S_{C_\bullet}\rangle$. The transitions connected to these two places are Put and Get. So we have to check whether the following formulas simplify into $null_{C_\bullet}$:
Put: $P2a[\text{buffer}] \circ W^+(\text{Put}, \text{buffer}) - P2a[\text{empty}] \circ W^-(\text{Put}, \text{empty}) = \langle S_{C_\bullet}\rangle \circ \langle X_1^1, X_2^1\rangle - \langle S_{C_\bullet}\rangle \circ \langle X_1^1, X_2^1, X_2^2\rangle = C_\bullet - C_\bullet = null_{C_\bullet}$.
Get: - $P2_a[\text{empty}] \circ W^+(\text{Get}, \text{empty})$ - $P2_a[\text{buffer}] \circ W^-(\text{Get}, \text{buffer}) = \langle S_{C_\bullet}\rangle \circ \langle S_{C_\bullet}\rangle[g] - \langle S_{C_\bullet}\rangle \circ \langle X_1^1, X_2^1, X_2^2\rangle[g] = C_\bullet - C_\bullet = nullC_\bullet$.

Let us check that the the transitions in each T-invariant produce a null change on all places, starting with T1.

Producing:
$W^+(\mathsf{Put,Producing}) \circ T1[\mathsf{Put}] - W^-(\mathsf{Produce, Producing}) \circ T1[\mathsf{Produce}] = \langle X_2^1 \rangle \circ |C_1| \langle X_2^1 \rangle - \langle X_2^1 \rangle \circ |C_1| \langle X_2^1 \rangle = \epsilon_{C_1}$ (the equations for Wait_to_insert, Wait_to_extract and Processing are very similar).
Wait_to_insert:
$W^+(\mathsf{Produce, Wait\_to\_insert}) \circ T1[\mathsf{Produce}] - W^-(\mathsf{Put, Wait\_to\_insert}) \circ T1[\mathsf{Put}] = null_{C_1}$ (same as above)
Wait_to_extract:
$W^+(\mathsf{End\_Process, Wait\_to\_extract}) \circ T1[\mathsf{End\_Process}]$ - $W^-(\mathsf{Get, Wait\_to\_extract}) \circ T1[\mathsf{Get}] = null_{C_1}$ (similar to above)
Processing:
$W^-(\mathsf{Get, Wait\_to\_extract}) \circ T1[\mathsf{Get}]$ -$W^+(\mathsf{End\_Process, Processing}) \circ T1[\mathsf{End\_Process}] = null_{C_1}$ (similar to above)
buffer:
$W^+(\mathsf{Put, buffer}) \circ T1[\mathsf{Put}] - W^-(\mathsf{Get, buffer}) \circ T1[\mathsf{Get}] = \langle X_1^1, X_2^1 \rangle \circ \langle S_{C_1}, X_2^1 \rangle - \langle X_1^1, X_2^1 \rangle [g] \circ \langle S_{C_1}, X_2^1, X_2^1 \rangle = \langle S_{C_1}, X_2^1 \rangle - \langle S_{C_1}, X_2^1 \rangle = \epsilon_{C_1,C_2}$
full:
$W^+(\mathsf{Put, full}) \circ T1[\mathsf{Put}] - W^-(\mathsf{Get, full}) \circ T1[\mathsf{Get}] = \langle S_{C_\bullet} \rangle \circ \langle X_1^1, X_2^1 \rangle - \langle S_{C_\bullet} \rangle [g] \circ \langle X_1^1, X_2^1, X_2^1 \rangle = \epsilon_{C_\bullet}$ (the equation for empty is very similar)
first:
$W^+(\mathsf{Get, first}) \circ T1[\mathsf{Get}] - W^-(\mathsf{Get, first}) \circ T1[\mathsf{Get}] = \langle !X_1^1 \rangle [g] \circ \langle S_{C_1}, X_2^1, X_2^1 \rangle - \langle X_1^1 \rangle [g] \circ \langle S_{C_1}, X_2^1, X_2^1 \rangle = \langle S_{C_1} \rangle - \langle S_{C_1} \rangle = \epsilon_{C_1}$ (the equation for last is very similar)

T2 has color domain C_1, C_2^8 where $8 = 2|C_1|$, completed by guard g' (see Table1); and similar formulae as for T1 apply to it, let us consider the expressions for just a few places to illustrate the type of composition to be solved.

buffer:
$W^+(\mathsf{Put, buffer}) \circ T2[\mathsf{Put}] - W^-(\mathsf{Get, buffer}) \circ T2[\mathsf{Get}] = \langle X_1^1, X_2^1 \rangle \circ (\langle X_1^1, X_2^1 \rangle [g'] + \langle !X_1^1, X_2^3 \rangle [g'] + \langle !^2 X_1^1, X_2^5 \rangle [g'] + \langle !^3 X_1^1, X_2^7 \rangle [g']) - \langle X_1^1, X_2^1 \rangle [g] \circ (\langle X_1^1, X_2^1, X_2^2 \rangle [g'] + \langle !X_1^1, X_2^3, X_2^4 \rangle [g'] + \langle !^2 X_1^1, X_2^5, X_2^6 \rangle [g'] + \langle !^3 X_1^1, X_2^7, X_2^8 \rangle [g']) = \epsilon_{C_1,C_2}$
Wait_to_insert:
$W^+(\mathsf{Produce, Wait\_to\_insert}) \circ T2[\mathsf{Produce}] - W^-(\mathsf{Put, Wait\_to\_insert}) \circ T2[\mathsf{Put}] = \langle X_2^1 \rangle \circ (\langle X_2^1 \rangle + \langle X_2^3 \rangle + \langle X_2^5 \rangle + \langle X_2^7 \rangle)[g'] - \langle X_2^1 \rangle \circ (\langle X_1^1, X_2^1 \rangle [g'] + \langle !X_1^1, X_2^3 \rangle [g'] + \langle !^2 X_1^1, X_2^5 \rangle [g'] + \langle !^3 X_1^1, X_2^7 \rangle [g']) = \epsilon_{C_2}$

Observe that in all cases where transition Get is involved we have an infix guard g whose terms check which static sublcass X_2^1 and X_2^2 belong to: it is easily handled both in T1 (where the involved elements of the right tuple are equal) and in T2 (where the T-semiflow guard g' implies that g is satisfied).
Finally, note that depending on the actual binding of the T-psemiflow variables to colors, functions $T_j[\mathsf{Produce}]$ and $T_j[\mathsf{End\_Process}]$ map to a $Bag^+[C_2]$ that may have coefficients greater than one. Hence the composition presented in this report is required to verify the invariant.

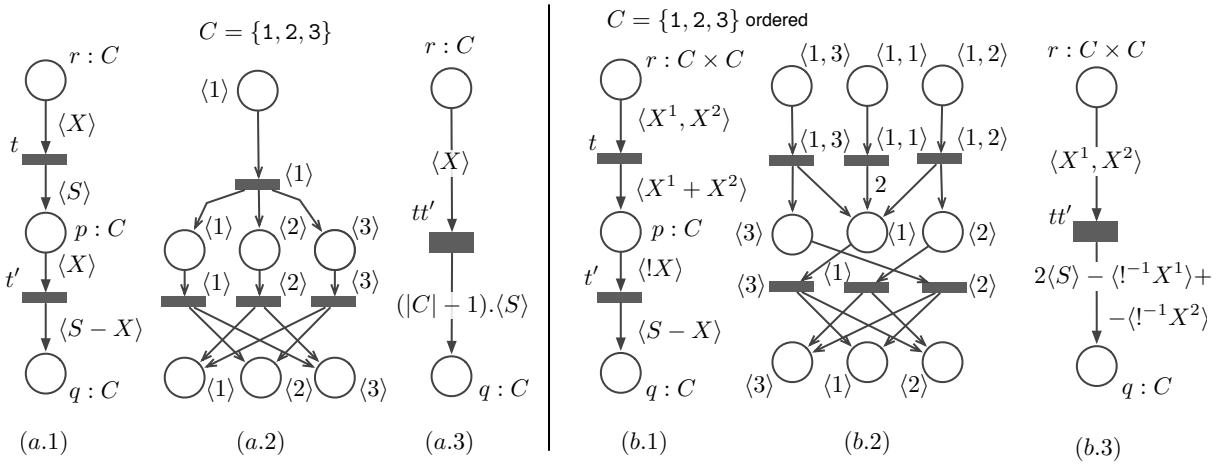

Figure 2. Agglomeration: .1 Original SN; .2 Unfolded subnet; .3 Reduced folded net.

Reduction of transitions Another application of multiset composition in coloured PN is structural reduction. For instance, in [11] a symbolic reduction technique (based on the capability to syntactically solve the composition) is applied to models used for software verification. To allow for a symbolic treatment, the authors propose Quasi-well formed Nets, a tight restriction of SNs. Moreover, the syntactical arc function composition requires further restrictions on the shape of manageable arc functions.

Transitions agglomeration is a useful reduction: it consists of merging causally connected transitions. The aim is to reduce transition instances interleaving, preserving specific system properties. A number of techniques have been proposed, with different properties and applicability conditions. Common to most proposals is this applicability scenario: a set H of transitions put tokens into a place p, leading to a new marking of p from which it is possible to restore its initial state *only* through the firing of a newly enabled transition t'; in that case, the firing of any transition in H may immediately cause the firing of t', without interfering with any other transition firing. In this report, we do not propose a general agglomeration technique for SN, rather, we illustrate how to perform agglomeration by symbolically deriving the arc functions for the new transition. We also provide an intuitive structural condition on SN arc functions, which ensures the applicability of agglomeration in a limited set of cases.

Figure 2(a.1) shows a simple example where the transitions t and t' can be agglomerated. Figure 2(a.2) depicts a part of the net unfolding connected to the instance $\langle 1 \rangle$ of t. It is possible to see that, if p is initially empty, after the firing of instance $\langle 1 \rangle$ of t, we can safely fire immediately all the newly enabled instances of t' restoring the p empty state. This is true for any instance $\langle i \rangle$ of t. Figure 2(a.3) is the reduced colored subnet, where the size of class C is a parameter.

The reduced subnet can be obtained without unfolding it: indeed, for this type of agglomeration, the following formula allows to symbolically compute the arc functions of the reduced subnet[8]:

$$W^-(tt', r) = W^-(t, r); \quad W^+(tt', q) = W^+(t', q) \circ W^-(t', p)^t \circ W^+(t, p)$$

[8] $W^-(t', p)^t : \mathcal{C}(p) \to Bag[\mathcal{C}(t)]$ denotes the transpose of $W^-(t', p)$; the rules to symbolically compute the transpose of an arc function are defined in [4].

In $W^+(tt', q)$ the rightmost composition provides the instances of t' enabled by the firing of an instance of t, through p. Finally composing $W^+(t', q)$ with this result (firing t' for all those instances) we obtain a function providing the multiset of tokens to be added in q. For the example of Fig.2(a) $W^-(tt', r) = \langle X \rangle$, while:

$$W^+(tt', q) = \langle S - X \rangle \circ \langle X \rangle^t \circ \langle S \rangle = \langle S - X \rangle \circ \langle S \rangle = (|C| - 1).\langle S \rangle \overset{|C|=3}{=} 2.\langle S \rangle$$

Figure 2(b.1) shows another example of reduction. The formula for the agglomeration can be also used here. The agglomeration is valid for any $|C| \geq 2$: $W^+(tt', q) = \langle S - X \rangle \circ \langle !X \rangle^t \circ \langle X^1 + X^2 \rangle = \langle S - X \rangle \circ \langle !^{-1}X \rangle \circ \langle X^1 + X^2 \rangle = \langle S - X \rangle \circ \langle !^{-1}X^1 + !^{-1}X^2 \rangle = \langle S - !^{-1}X^1 \rangle + \langle S - !^{-1}X^2 \rangle = 2.\langle S \rangle - \langle !^{-1}X^1 \rangle - \langle !^{-1}X^2 \rangle$

For these examples where p is the only input place of t' and is the only output place of t, we can confidently state that if (1) the instances of transition t' are not in conflict with each other (a situation called *auto-conflict*, which can be checked symbolically on the net structure) and (2) if all the tokens put into p by t can be completely consumed by (one or more instances of) t', then we can aggregate t and t'. To verify the second condition it is sufficient to check the following equality: $W^+(t, p) = W^-(t', p) \circ W^-(t', p)^t \circ W^+(t, p)$. The following property characterizes the form for the input function $W^-(t', p)$ which guarantees the agglomeration conditions(s). Let the identity on D be $Ide_D(d) = 1 \cdot d$, $\forall d \in D$.

Property 18. Let $\mathcal{C}(p)' = \overline{W^+(t, p)(\mathcal{C}(t))}$ be the (support of the) image of $W^+(t, p)$.

$$W^+(t, p) = W^-(t', p) \circ W^-(t', p)^t \circ W^+(t, p) \Leftrightarrow \forall c \in \mathcal{C}(p)' \; W^-(t', p) \circ W^-(t', p)^t(c) = 1 \cdot c$$

(i.e. the restriction of $W^-(t', p) \circ W^-(t', p)^t$ to $\mathcal{C}(p)'$ is the identity).

Observe that with the assumption that p is the unique intermediate place between t and t', this property includes also the condition of no autoconflicts among t' instances ($= \overline{W^-(t', p)^t \circ W^-(t', p)} - \overline{Ide}$, simplified formula from [5]).

The condition holds for both the examples discussed above, and this can be checked symbolically exploiting the composition presented in this report.
Finally, let us consider a third example, fully exploiting the composition technique presented in this report. The subnet schema is the same but for t', which has one more output place, q'. The arc functions for this example are:

$$W^-(t, r) = \langle X \rangle \quad W^+(t, p) = \langle S - !X, S, S - X \rangle \quad W^+(t', q) = \langle !X^2, X^2 \rangle [X^1 = X^3]$$
$$W^-(t', p) = \langle !X^3, X^2, X^1 \rangle \qquad W^+(t', q') = 2 \cdot \langle X^1, X^2 \rangle [X^1 \neq X^2 \wedge X^2 \neq X^3]$$

The arc functions for the agglomerated transition tt' are ($\lambda = |C|$ with $|C| > 2$):

$$W^-(tt', r) = \langle X \rangle \qquad W^+(tt', q) = (\lambda - 1)[X^1 = !X^2]\langle S, S \rangle$$
$$W^+(tt', q') = 2(\lambda - 1)\langle S - X, X \rangle + 2(\lambda - 2)[X^1 \neq X^2]\langle S - X, S - X \rangle$$

The details of the derivation of $W^+(tt', q')$ follows, with the indication of the involved Lemma in each step. Let $\lambda = |C|$ ($|C| > 2$): $W^+(tt', q') = W^+(t', q') \circ W^-(t', p)^t \circ W^+(t, p)$.

$$
\begin{aligned}
W^+(tt', q') &= 2 \cdot \langle X^1, X^2 \rangle [X^1 \neq X^2 \wedge X^2 \neq X^3] \circ \langle !X^3, X^2, X^1 \rangle^t \circ \langle S - !X, S, S - X \rangle \\
&\overset{\text{transpose}}{=} 2 \cdot \langle X^1, X^2 \rangle [X^1 \neq X^2 \wedge X^2 \neq X^3] \circ \langle X^3, X^2, !^{-1}X^1 \rangle \circ \langle S - !X, S, S - X \rangle \\
&\overset{\text{Lemma 4.2}}{=} 2 \cdot \langle X^1, X^2 \rangle [X^1 \neq X^2 \wedge X^2 \neq X^3] \circ \langle S - X, S, S - X \rangle \\
&\overset{\text{Definition 4.1 (Claim 1)}}{=} 2 \cdot \langle X^1, X^2 \rangle [X^1 \neq X^2 \wedge X^2 \neq X^3] \circ \langle S - X, S - X, S - X \rangle \\
&+ 2 \cdot \langle X^1, X^2 \rangle \circ \langle S - X, X, S - X \rangle \\
&\overset{\text{Corollary 4.2, Lemma 4.4}}{=} 2 \cdot \langle X^1, X^2 \rangle (\lambda - 2)[X^1 \neq X^2] \circ \langle S - X, S - X \rangle + 2(\lambda - 1)\langle S - X, X \rangle \overset{\text{Lemma 4.1, 4.2}}{} \\
&\overset{\text{Lemma 4.3}}{=} 2(\lambda - 2)[X^1 \neq X^2]\langle S - X, S - X \rangle + 2(\lambda - 1)\langle S - X, X \rangle
\end{aligned}
$$

6. Conclusions and future work

This report reaches the goal of completing the symbolic manipulation of SN arc functions defining a procedure for the composition of functions mapping on Bags. This enables a number of applications that couldn't be treated with the operations defined so far. The implementation of a new version of SNexpression including this kind of composition is ongoing. The approach suggests an extension to the SN formalism itself, to allow filters in arc expressions: this feature is now implemented in the GreatSPN Graphical User Interface. The presented results are often parametric in color class cardinality, possibly with some constraints on cardinality lower bounds.

Future work is devoted to extend the possible applications of the results presented in this report, in particular as concerns a generalization of net reduction methods proposed in literature to a wider class of nets: for instance, we expect that our proposal may allow to relax some restriction posed in [11]. Another interesting topic is the elimination of immediate transitions in Stochastic SNs (to be able to apply analysis algorithms which do not work in presence of immediate transitions): this is a challenging problem since in general it may involve marking dependent sequences of immediate transition firings triggered by one timed transition, as in the case of the example in Fig. 1.

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A. Appendix

In this appendix the reader can find formal proofs of the Properties and Lemmas presented in the report, some relevant guard/filter reduction rules, generalization of Case 3 (Sec. 4), and a complete example of composition procedure application.

A.1. Proof of Properties and Lemmas

Proof of Property 3.: *composition of a constant and a constant-size function*.
The equality is trivially verified for any $a : g(a) = false$. Consider a: $g(a) = true$. Let $c \in Bag[D]$ be the image of f: $f \circ h(a) = \sum_{x \in h(a)} h(a)(x) \cdot c = n \cdot c$. □
Proof of Property 4: *composition of a tuple including a constant*.
Let $c \in Bag[D]$ be the image of f', we have:
$\langle f, f' \rangle \circ h(a) = \sum_{x \in h(a)} h(a)(x) \cdot \langle f(x), c \rangle$. By exchanging the sum and the product we get
$\langle \sum_{x \in h(a)} h(a)(x) f(x), c \rangle = \langle f \circ h, f' \rangle(a)$. □
Proof of Lemma 4.1: *elimination of right tuple elements not in the image of* $\overline{X}$.
Let $T' = \langle T'_{\overline{X}}, T'_{\neg \overline{X}} \rangle$. Based on Property, 10 it sufficient to show that:
$T'_{\neg \overline{X}}[g']$ constant-size, $|T'_{\neg \overline{X}}| = k \Leftrightarrow \forall a \in D''\ g'(a) \Rightarrow \mathbf{\Pi}_{\overline{X}} \circ T'(a) = k \cdot T'_{\overline{X}}(a)$.
Let $a \in D''$, $g'(a) = true$. Then $T'(a) = \langle T'_{\overline{X}}(a), T'_{\neg \overline{X}}(a) \rangle$ and $T(T'(a)) = \sum_{a_1 \in T'_{\overline{X}}(a), a_2 \in T'_{\neg \overline{X}}(a)} T'_{\overline{X}}(a)(a_1) \cdot T'_{\neg \overline{X}}(a)(a_2) T(\langle a_1, a_2 \rangle)$. Since $T(\langle a_1, a_2 \rangle) = T^r(a_1)$, by replacing in $T(T'(a))$ we get:
$\sum_{a_1 \in T'_{\overline{X}}(a), a_2 \in T'_{\neg \overline{X}}(a)} T'_{\overline{X}}(a)(a_1) \cdot T'_{\neg \overline{X}}(a)(a_2) T^r(a_1)$.
Rewriting it as: $\sum_{a_1 \in T'_{\overline{X}}(a)} T'_{\overline{X}}(a)(a_1) \cdot (\sum_{a_2 \in T'_{\neg \overline{X}}(a)} T'_{\neg \overline{X}}(a)(a_2)) \cdot T^r(a_1)$, since the inner sum is k (Hp), the whole expression becomes $k \cdot T^r \circ T'_{\overline{X}}(a)$. □
Proof of Lemma 4.2: *composition of independent subtuples*.
Without loss of generality, let $h = 2$, $T' = \langle T'_1, T'_2 \rangle$, $T'_1 = T'_{\overline{X}_1}$, $T'_2 = T'_{\overline{X}_2}$. Consider $a \in D''$.
$T(T'(a)) = \sum_{a_1 \in T'_1(a), a_2 \in T'_2(a)} T'_1(a)(a_1) \cdot T'_2(a)(a_2) T(\langle a_1, a_2 \rangle)$. $T(\langle a_1, a_2 \rangle) = \langle F_1(\langle a_1, a_2 \rangle), F_2(\langle a_1, a_2 \rangle \rangle$
$= \langle {F_1}^r(a_1), {F_2}^r(a_2) \rangle$.
Replacing in $T(T'(a))$ we get:
$\sum_{a_1 \in T'_1(a), a_2 \in T'_2(a)} T'_1(a)(a_1) \cdot T'_2(a)(a_2) \langle {F_1}^r(a_1), {F_2}^r(a_2) \rangle = \langle {F_1}^r \circ T'_1, {F_2}^r \circ T'_2 \rangle(a)$. □
Proof of Property 11: *constant size simple tuple-prefix*.
Let $T = \otimes_r f_r$. It directly comes from the fact that for any two color-tuples $c, c' \in D$, and for any clause $eq : X^i \neq^{!s} X^j$ in g, we have: $f_i(c) =^{!s} f_j(c)$, $f_i(c') =^{!s} f_j(c')$, and $|f_i(c)| = |f_i(c')|$, i.e., the number of color-tuples which satisfy g is the same in $T(c)$ and $T(c')$. □
Proof of Property 13: *projection repetition (unordered class)*.
$\langle X^1, \ldots, X^1 \rangle \circ h(a) = \sum_{a_i \in h(a)} \langle a_i, \ldots, a_i \rangle = [\bigwedge_{i:2 \ldots m} X^1 = X^i] \langle h, \ldots, h \rangle(a)$. □
Proof of Property 15: *projection repetition with one occurrence of* $S - X^1$.
Let E_r, E_l be the right- and left-hand expression, respectively.
$E_r(a) = [\ldots] \langle h(a), \ldots, h(a), S \rangle \stackrel{Hp}{=} sum_{a' \in h(a)} \langle a', \ldots, a', S - a' \rangle = E_l(a)$. □
Proof of Property 16: *one projection with* $S - X^1$.
$[X^1 \neq X^2] \langle h, S \rangle(a) = \sum_{a' \in h(a)} h(a)(a') \langle a', S - a' \rangle = \langle X^1, S - X^1 \rangle \circ \langle h(a) \rangle$. □
Proof of Corollary 4.2: *Infix predicate elimination (case 3:* $\overline{X} \subset Var(g)$*)*.

The first equality is just a term substitution (property 10). As for k, it follows from $|\mathbf{\Pi}_{\overline{X}} \circ [g]T'| = |[g]T'|$ (given that $[g]T'$ is constant-size also its projection is, accordingly) and from $[g_{\overline{X}}]^r T'_{\overline{X}}$ being constant-size by construction. □

Proof of Property 18: *transition agglomeration condition.*
$\Leftarrow$ is obvious. As for $\Rightarrow$, assume that there exists $x \in \mathcal{C}(t)$ such that $W^+(t,p)(x) = b \neq \epsilon$, for which it holds $W^-(t',p) \circ W^-(t',p)^t(b) \neq b$. We get a contradiction. □

A.2. Filter base reduction rules

The rules below, when recursively applied, make filters prefixing tuples meet the 1st condition of Definition 4.1. Symbols $\cap, \ominus$ (used only in this context) denote set-intersection/difference, e, f any elementary/class-function, respectively. We assume that $!^h X^i = !^k X^i \overset{h \neq k}{\rightarrow} false$.
Elementary intersections/differences.

$$S \cap e \rightarrow e \qquad S_i \cap S_j \overset{i \neq j}{\rightarrow} \emptyset \qquad X^i \cap S_i \rightarrow X^i[X^i \in C_i] \qquad !^h X^i \cap !^k X^j \rightarrow !^h X^i[X^i = !^{h-k} X^j]$$
$$e \ominus S \rightarrow \emptyset \qquad S_i \ominus S_j \overset{i \neq j}{\rightarrow} S_i \qquad X^i \ominus S_i \rightarrow X^i[X^i \notin C_i] \qquad !^h X^i \ominus !^k X^j \rightarrow !^h X^i[X^i \neq !^{h-k} X^j]$$
$$S \ominus e \rightarrow S - e$$

Reduction of filter predicates (in decreasing order of priority)

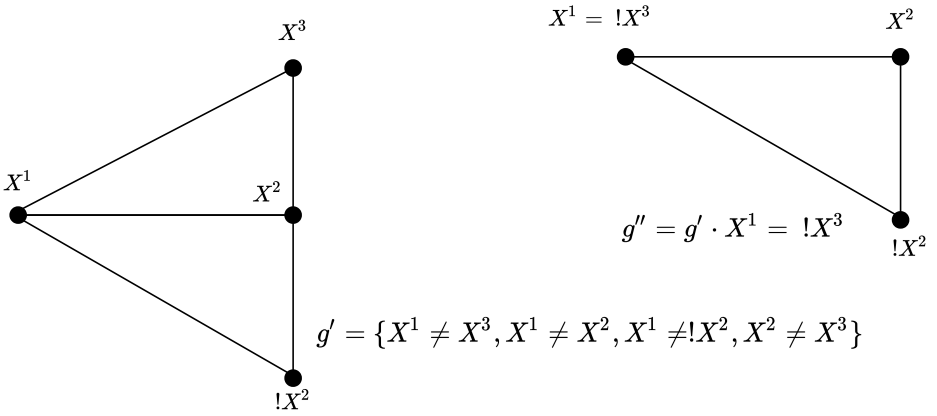
$$[\ldots, X^i \in C_k]\langle \ldots, \sum_j \underset{i}{\lambda_j} e_j, \ldots\rangle \rightarrow \qquad [\ldots]\langle \ldots, \sum_j \underset{i}{\lambda_j} (e_j \cap S_k), \ldots\rangle$$
$$[\ldots, X^i \notin C_k]\langle \ldots, \sum_j \underset{i}{\lambda_j} e_j, \ldots\rangle \rightarrow \qquad [\ldots]\langle \ldots, \sum_j \underset{i}{\lambda_j} (e_j \ominus S_k), \ldots\rangle$$
$$[X^i = !^k X^j, \ldots]\langle \ldots, \underset{i}{e}, \ldots, \sum_h \underset{j}{\lambda_h} e_h, \ldots\rangle \overset{|e|=1}{\rightarrow} \qquad [\ldots]\langle \ldots, \underset{i}{e}, \ldots, \sum_h \underset{j}{\lambda_h} (e_h \cap !^{-k} e), \ldots\rangle$$
$$[X^i \neq !^k X^j, \ldots]\langle \ldots, \underset{i}{e}, \ldots, \sum_h \underset{j}{\lambda_h} e_h, \ldots\rangle \overset{|e|=1}{\rightarrow} \qquad [\ldots]\langle \ldots, \underset{i}{e}, \ldots, \sum_h \underset{j}{\lambda_h} (e_h \ominus !^{-k} e), \ldots\rangle$$

$$[X^i = !^k X^j, \ldots]\langle \ldots, \underset{i}{e}, \ldots, \underset{j}{e'}, \ldots\rangle \overset{|e|>1 \wedge |e'|>1 \wedge e \not\equiv !^k e'}{\rightarrow} \qquad [X^i = !^k X^j]\langle \ldots, e \cap \underset{i}{!^k e'}, \ldots, e' \cap \underset{j}{!^{-k} e}, \ldots\rangle$$

$$[X^i \neq !^k X^j, \ldots]\langle \ldots, \underset{i}{e}, \ldots, \underset{j}{e'}, \ldots\rangle \overset{|e|>1 \wedge |e'|>1 \wedge e \not\equiv !^k e'}{\rightarrow} \qquad [X^i \neq !^k X^j]\langle \ldots, e \cap \underset{i}{!^k e'}, \ldots, e' \cap \underset{j}{!^{-k} e}, \ldots\rangle +$$
$$[\ldots]\langle \ldots, e \ominus \underset{i}{!^k e'}, \ldots, \underset{j}{e'}, \ldots\rangle +$$
$$[\ldots]\langle \ldots, \underset{i}{e}, \ldots, e' \ominus \underset{j}{!^{-k} e}, \ldots\rangle$$

$$[X^i \overset{\not\equiv}{=} !^k X^j, \ldots]\langle \ldots, \sum_k \underset{i}{\lambda_k} e_k, \ldots, \sum_h \underset{j}{\lambda'_h} e'_h, \ldots\rangle \rightarrow \qquad \sum_{k,h} \lambda_k \lambda'_h [X^i \overset{\not\equiv}{=} !^k X^j, \ldots]\langle \ldots, \underset{i}{e_k}, \ldots, \underset{j}{e'_h}, \ldots\rangle$$

Remark. In rules 3 through 6, we may actually replace (elementary) symbols e, e' with any *type-set* (considering also the tuple guard) class-functions $f = \sum_h \lambda_h e_h$, $f' = \sum_k \lambda'_k e'_k$. We have, $f \cap f' = \sum_{h,k} \lambda_h \lambda'_k e_h \cap e'_k$, and $f \ominus f' = f - f \cap f'$.



$$Sol(g' + X^1 \neq !X^3, \lambda) = \lambda(\lambda-2)^2 - \lambda(\lambda-2) = \lambda(\lambda-2)(\lambda-3)$$

Figure 3. An example of incremental computation of $Sol(g, \lambda)$

A.3. Examples of Case 3 with ordered classes

Consider the following composition, meeting Definition 4.1. It is a representative case, because when an ordered class is involved we can always reduce to simple tuple-prefixes where the right tuple T' contains uniquely the diffusion-function S. Note that g cannot be expressed so that $|Var(g)| = |Symb(g)|$.

$$\langle X^1, X^2\rangle[X^1 \neq X^2 \wedge X^1 \neq X^3 \wedge X^2 \neq !^{-1}X^3] \circ \langle S, S, S\rangle \;\; |C| > 2$$

However, we may conveniently rewrite it as:

$$\langle \underset{T}{X^1, X^2}\rangle[X^1 \neq X^2 \wedge X^1 \underset{g}{\neq} X^3 \wedge X^3 \neq !X^2] \circ \langle \underset{T'}{S, S, S}\rangle \qquad |C| > 2$$

In this equivalent form, any inequality between $Var(T)$ variables $\overline{X} = \{X^1, X^2\}$ and X^3 involves the same symbol (X^3). This makes it possible to apply Lemma 4.5. The sub-graph $g_{\overline{X}}$, $\overline{X} = \{X^1, X^2\}$, whose vertices are $\{X^1, X^2, !X^2\}$, becomes a "clique" by linking X^1 to $!X^2$, considering the implicit inequality $X^2 \neq !X^2$.

$$\begin{aligned} T[g]T' &\equiv \langle X^1, X^2\rangle[g \wedge X^1 \neq !X^2] \circ T' + \langle X^1, !^{-1}X^1\rangle[X^1 \neq X^3] \circ T' \\ &\equiv \langle X^1, X^2\rangle \circ (\lambda-2)[X^1 \neq X^2 \wedge X^1 \neq !X^2]\langle S, S\rangle + \langle X^1, !^{-1}X^1\rangle \circ (\lambda-1)\langle S\rangle \\ &\equiv (\lambda-2)[X^1 \neq X^2 \wedge X^1 \neq !X^2]\langle S, S\rangle + (\lambda-1)[X^1 = !X^2]\langle S, S\rangle \end{aligned}$$

$\lambda - 2$ is the ratio $\frac{\lambda(\lambda-2)^2}{\lambda(\lambda-2)}$, where numerator and denominator refer to $g' := g + X^1 \neq !X^2$ and its restriction to $\overline{X}$ (the clique $\{X^1, X^2, !X^2\}$), respectively. Analogously, $\lambda - 1$ is the ratio $\frac{\lambda(\lambda-1)}{\lambda}$, where numerator and denominator refer to $g'' := g \cdot X^1\ !X^2$ (whose simple form results in $X^1 \neq X^3$, $X^1 = !X^2$) and its restriction to $\overline{X}$ (the 1-order empty graph $\{X^1\}$), respectively.

For the computation of $Sol(g, \lambda)$, it may be convenient to rephrase the fundamental reduction theorem used for the computation of $P(g, \lambda)$ as follows: let $x, y \in Var(g)$ and $e := x \neq !^r y$, $e \notin g$: $Sol(g + e, \lambda) = Sol(g, \lambda) - Sol(g \cdot x\ !^r y, \lambda)$, where $g \cdot x\ !^r y$ is the graph isomorphic to the predicate

obtained from g by replacing the occurrences of x with $!^r y$, and removing redundant edges: indeed, $x = !^r y \Rightarrow x \neq !^s y, \forall s \neq r$ (assuming $|s - r| < |C|$).

Fig. 3 shows another application of the reduction theorem above: the goal is to compute $Sol(g, \lambda)$ where $g = [g' \wedge X^1 \neq !X^3]$, given that $Sol(g', \lambda) = \lambda(\lambda - 2)^2$ and $Sol(g'', \lambda) = \lambda(\lambda - 2)$ are known.

A.4. Example of a complete composition procedure

Fig. 4 shows the main steps of a complete composition procedure where most of properties and lemmas apply: C_1 is a partitioned class, C_2 is ordered.

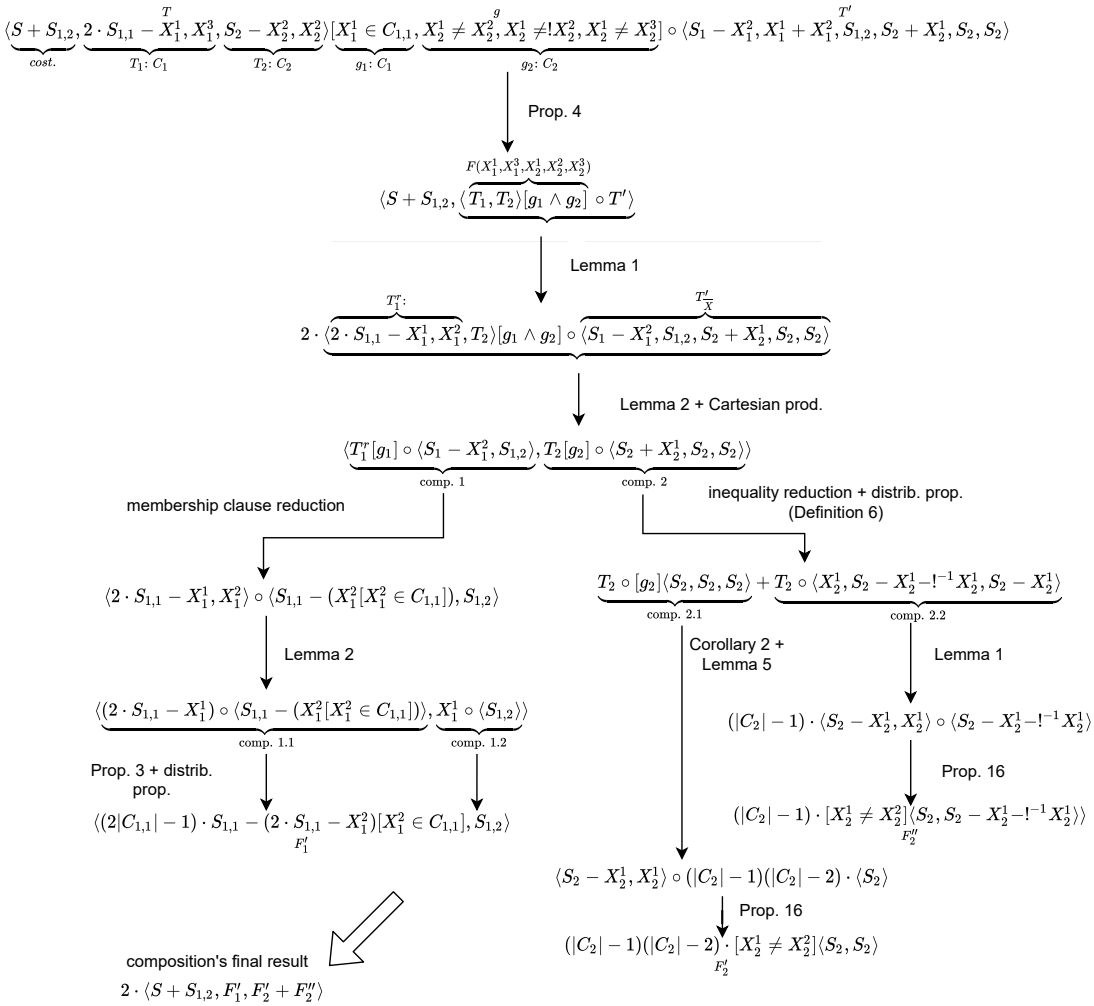


Figure 4. A complete composition example