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Reasoning about exceptions in ontologies: from the lexicographic closure to the skeptical closure

L. Giordano, V. Gliozzi (laura.giordano@uniupo.it, gliozzi@di.unito.it)

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Laura Giordano
DISIT, Università del Piemonte Orientale “A. Avogadro”
Viale Teresa Michel, 11 - 15121, Alessandria, Italy
laura.giordano@uniupo.it

Valentina Gliozzi
Center for Logic, Language and Cognition
Dipartimento di Informatica, Università di Torino,
Corso Svizzera, 185, 10149 Torino, Italy,
valentina.gliozzi@unito.it

Abstract

Reasoning about exceptions in ontologies is nowadays one of the challenges the description logics community is facing. The paper describes a preferential approach for dealing with exceptions in Description Logics, based on the rational closure. The rational closure has the merit of providing a simple and efficient approach for reasoning with exceptions, but it does not allow independent handling of the inheritance of different defeasible properties of concepts. In this work we outline a possible solution to this problem by introducing a weaker variant of the lexicographical closure, that we call *skeptical closure*, which requires to construct a single base. We develop a bi-preference semantics for defining a characterization of the skeptical closure.

1 Introduction

Reasoning about exceptions in ontologies is nowadays one of the challenges the description logics community is facing, a challenge which is at the very roots of the development of non-monotonic reasoning in the 80s. Many non-monotonic extensions of Description Logics (DLs) have been developed incorporating non-monotonic features from most of the non-monotonic formalisms in the literature [59, 3, 26, 28, 46, 16, 10, 38, 21, 54, 27, 9, 48, 24, 19, 39, 40], or defining new constructions and semantics such as [45, 8, 12].

We focus on the rational closure for DLs [21, 19, 41, 40, 18] and, in particular, on the construction developed in [40], which is semantically characterized by minimal (canonical) preferential models. While the rational closure provides a simple and efficient approach for reasoning with exceptions, exploiting polynomial reductions to standard DLs [33, 53, 36, 14], the rational closure does not allow an independent handling of the inheritance of different defeasible properties of concepts[1] so that, if a subclass of C is exceptional for a given aspect, it is exceptional tout court and does not inherit any of the typical properties of C. This problem was called by Pearl [56] the "blockage of property inheritance" problem, and it is an instance of the "drowning problem" in [6].

To cope with this problem Lehmann [51] introduced the notion of the lexicographic closure, which was extended to Description Logics by Casini and Straccia [23], while in [24] the same authors develop an inheritance-based approach for defeasible DLs. Other proposals to deal with this "all or nothing" behavior in the context of DLs are the Relevant Closure [17] by Casini et al., the logic of overriding, $\mathcal{DL}^N$, by Bonatti,et al. [8, 11], a nonmonotonic description logic in which conflicts among defaults are solved based on specificity, and the work by Gliozzi [44], who develops a multi-preference semantics for defeasible inclusions in which models are equipped with several preference relations. The idea of having different preference relations was first proposed by Gil [29] to define a multi-typicality extension of $\mathcal{ALC}$+$\mathbf{T}_{min}$ [39], a logic with a different minimal model semantics w.r.t. the rational closure semantics.

In this paper we will consider a variant of the lexicographic closure. The lexicographic closure allows for stronger inferences with respect to rational closure, but computing the defeasible consequences in the lexicographic closure may require to compute several alternative *bases* [51], namely, consistent sets of defeasible inclusions which are maximal with respect to the (so called) seriousness ordering. We propose an alternative notion of closure, the *skeptical closure*, which can be regarded as a more skeptical variant of the lexicographic closure, which does not require to generate alternative maximally consistent bases for a given concept. Roughly speaking, to check the defeasible properties of a concept C, the construction builds a single maximal consistent set of defeasible inclusions compatible with C (a base for C), starting from the defeasible inclusions with highest rank and progressively adding less specific inclusions, when consistent. If there are conflicting defeasible inclusions at a certain stage, all defeasible inclusions with equal or lower rank are excluded. Our construction requires a polynomial number of calls to an underlying preferential $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ reasoner to establish the defeasible properties of a concept C.

To develop a semantic characterization of the skeptical closure, we introduce a bi-preference semantics (BP-semantics), which is still in the realm of the preferential semantics for defeasible description logics [37, 16, 38], developed along the lines of the preferential semantics by Kraus, Lehmann and Magidor [49, 50]. The BP-semantics has two preference relations and is a refinement of the rational closure semantics. We show that the BP-semantics provides a characterization of the MP-closure, a variant of the lexicographic closure introduced for $\mathcal{ALC}$ in [34, 32] as a sound approximation of the multipreference semantics. Using this semantic characterization, we show that the skeptical closure is well-behaved, as it satisfies all the KLM properties of a preferential consequence relation [49], and that it is neither weaker nor stronger than the Relevant

[1]By *properties* of a concept, here we generically mean characteristic features of a class of objects (represented by a set of inclusion axioms) rather than roles (properties in OWL [55]).

closure.

Plan of the paper is the following. Section 2 recalls the definition of the rational closure for $\mathcal{ALC}$ in [40] and of its semantics. Section 3 defines the skeptical closure. Section 4 introduces the bi-preference semantics and Section 5 shows that it provides a semantic characterization of the MP-closure for $\mathcal{ALC}$. In Section 6, the BP-semantics is used to define a semantic characterization for the skeptical closure, its KLM properties are studied, and comparisons with the Relevant Closure are given. Finally, in Section 7, we discuss related work and conclude the paper.

This work is based on the extended abstract presented at CILC/ICTCS 2017 [30], where the notion of skeptical closure was first introduced.

2 The rational closure for $\mathcal{ALC}$

We briefly recall the logic $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$ which is at the basis of a rational closure construction proposed in [40] for $\mathcal{ALC}$. The idea underlying $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$ is that of extending the standard $\mathcal{ALC}$ with concepts of the form $\mathbf{T}(C)$, whose intuitive meaning is that $\mathbf{T}(C)$ selects the *typical* instances of a concept C, to distinguish between the properties that hold for all instances of concept C ($C \sqsubseteq D$), and those that only hold for the typical such instances ($\mathbf{T}(C) \sqsubseteq D$). Given a set N_I of individual names, a set N_C of concept names, and a set N_R of role names, the $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$ language is defined as follows:

$$C_R := A \mid \top \mid \bot \mid \neg C_R \mid C_R \sqcap C_R \mid C_R \sqcup C_R \mid \forall R.C_R \mid \exists R.C_R$$
$$C_L := C_R \mid \mathbf{T}(C_R),$$

where $A \in N_C$ is a concept name and $R \in N_R$ a role name. A knowledge base K is a pair $(\mathcal{T}, \mathcal{A})$, where the TBox $\mathcal{T}$ contains a finite set of concept inclusions $C_L \sqsubseteq C_R$, and the ABox $\mathcal{A}$ contains a finite set of assertions of the form $C_R(a)$ and $R(a,b)$, for $a, b \in N_I$ and $R \in N_R$. The inclusions of the form $C_R \sqsubseteq C_R$ are called *strict*, and the set of the strict inclusions in $\mathcal{T}$ is denoted by $Strict_{\mathcal{T}}$. We call C_L an *extended* concept and C_R an $\mathcal{ALC}$ concept (or *non-extended* concept).

The semantics of $\mathcal{ALC}$ with typicality is defined in terms of preferential models, extending to $\mathcal{ALC}$ the preferential semantics by Kraus, Lehmann and Magidor in [49, 50]: ordinary models of $\mathcal{ALC}$ are extended with a *preference relation* $<$ on the domain Δ, whose intuitive meaning is to compare the "typicality" of domain elements: $x < y$ means that x is more typical than y. The instances of $\mathbf{T}(C)$ are the instances of concept C that are minimal with respect to $<$. The instances of a concept C are also called C-elements as they are the elements of the domain belonging to the interpretation of C. For a set S of domain elements we let $min_<(S) = \{u \mid u \in S$ and there is no z such that $z < u\}$ be the set of the minimal elements in S w.r.t. $<$. The preference relation $<$ is assumed to be *well-founded* (i.e., there is no infinite $<$-descending chain, so that, if $S \neq \emptyset$, also $min_<(S) \neq \emptyset$). In ranked models, which characterize $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$, $<$ is further assumed to be *modular* (i.e., for all $x, y, z \in \Delta$, if $x < y$ then either $x < z$ or $z < y$). Ranked models characterize $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$. Let us recap their definition.

Definition 2.1 (Preferential and ranked interpretations of $\mathcal{ALC}+\mathbf{T}$) *A preferential interpretation $\mathcal{M}$ is any structure $\mathcal{M} = \langle \Delta, <, I \rangle$ where: Δ is the domain; $<$ is an* *irreflexive, transitive and well-founded relation over* Δ*.* I *is an interpretation function that maps each concept name* $C \in N_C$ *to* $C^I \subseteq \Delta$*, each role name* $R \in N_R$ *to* $R^I \subseteq \Delta^I \times \Delta^I$ *and each individual name* $a \in N_I$ *to* $a^I \in \Delta$*. For concepts of* $\mathcal{ALC}$*,* C^I *is defined in the usual way in* $\mathcal{ALC}$ *interpetations [2], namely:* $\top^I = \Delta$*,* $\bot^I = \emptyset$*,* $(\neg C)^I = \Delta \backslash C^I$*,* $(C \sqcap D)^I = C^I \cap D^I$*,* $(C \sqcup D)^I = C^I \cup D^I$ *and*

$$(\forall R.C)^I = \{x \in \Delta \mid \textit{for all } y, (x, y) \in R^I \textit{ implies } y \in C^I\}$$
$$(\exists R.C)^I = \{x \in \Delta \mid \textit{for some } y\ (x, y) \in R^I \textit{ and } y \in C^I\}$$

For the $\mathbf{T}$ *operator, we have* $(\mathbf{T}(C))^I = min_<(C^I)$*. When the interpretation* I *is also* modular*,* I *is called a* ranked *interpretation.*

The notion of satisfiability of a KB in an interpretation $I = \langle \Delta, <, I\rangle$ is defined as usual:
- I satisfies an inclusion $C \sqsubseteq D$, if $C^I \subseteq D^I$;
- I satisfies an assertion $C(a)$ (resp., $R(a, b)$), if $a^I \in C^I$ (resp., $(a^I, b^I) \in R^I$).

In particular, I satisfies an inclusion $\mathbf{T}(C) \sqsubseteq D$ if $(\mathbf{T}(C))^I \subseteq D^I$ (i.e, if $min_<(C^I) \subseteq D^I$).

A preferential (ranked) model of a knowledge base $K = (\mathcal{T}, \mathcal{A})$ is a preferential (ranked) interpretation $\mathcal{M}$ that satisfies all inclusions in $\mathcal{T}$ and all assertions in $\mathcal{A}$. A *query* F (either an assertion $C_L(a)$ or an inclusion $C_L \sqsubseteq C_R$) is preferentially (rationally) entailed by a knowledge base K, i.e. $K \models_{\mathcal{ALC}+\mathbf{T}} F$ (resp., $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} F$), if F is satisfied in all the preferential (ranked) models of K. As an example of $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$ knowledge base consider the following:

Example 2.2 *Let* $K = (\mathcal{T}, \mathcal{A})$ *be a knowledge base with TBox* $\mathcal{T}$*:*

$$\mathbf{T}(Student) \sqsubseteq \neg Pay\_Taxes$$
$$\mathbf{T}(WStudent) \sqsubseteq Pay\_Taxes$$
$$\mathbf{T}(Student) \sqsubseteq Smart$$
$$WStudent \sqsubseteq Student$$

and ABox $\mathcal{A} = \{Student(tom), WStudent(mary), hasFriend(tom, mary)\}$*. The TBox states that typical students do not pay taxes, but typical working students (which are students) do pay taxes, and that typical students are smart. The ABox contains the individual names* tom *and* $mary$ *and the role name* $hasFriend$*. In this work, however, we will manly focus on defeasible reasoning involving TBox.*

The definition of the rational closure for $\mathcal{ALC}$ and its semantics in [40] exploit the extension $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$ of $\mathcal{ALC}$ with typicality, under a ranked semantics. As shown therein, $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$ enjoys the finite model property and finite $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$ models can be equivalently defined by postulating the existence of a function $k_\mathcal{M} : \Delta \longmapsto \mathbb{N}$, where $k_\mathcal{M}$ assigns a finite rank to each individual: the rank $k_\mathcal{M}(x)$ of a domain element $x \in \Delta$ is the length of the longest chain $x_0 < \cdots < x$ from x to a minimal x_0 (s.t. there is no x' with $x' < x_0$). The rank $k_\mathcal{M}(C_R)$ of a concept C_R in $\mathcal{M}$ is $i = min\{k_\mathcal{M}(x) : x \in C_R^I\}$.

Although the typicality operator $\mathbf{T}$ itself is nonmonotonic (i.e. $\mathbf{T}(C) \sqsubseteq D$ does not imply $\mathbf{T}(C \sqcap E) \sqsubseteq D$), the logic $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$ is monotonic: what is logically entailed by K is still entailed by any K' with $K \subseteq K'$. In [41, 40] a non-monotonic construction of rational closure (RC for short) has been defined for $\mathcal{ALC} + \mathbf{T}_R$, extending the construction of RC introduced by Lehmann and Magidor [50] to the description logic $\mathcal{ALC}$. The definition of RC is based on the notion of exceptionality. Roughly speaking $\mathbf{T}(C) \sqsubseteq D$ holds in the rational closure of $\mathcal{T}$ if C is less exceptional than $C \sqcap \neg D$. To determine the exceptionality of concepts and inclusions in RC, we build a non-increasing sequence $E_0 \supseteq E_1 \supseteq E_2 \ldots$ of subsets of $\mathcal{T}$, starting from $E_0 = \mathcal{T}$. We recall the construction from [40].

Definition 2.3 (Exceptionality of concepts and inclusions) *Let $E \subseteq \mathcal{T}$ and C a concept. C* is exceptional for *E if and only if $E \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg C$. An inclusion $\mathbf{T}(C) \sqsubseteq D$ is exceptional for E if C is exceptional for E. The set of inclusions which are exceptional for E will be denoted by $\mathcal{E}(E)$.*

Given a TBox $\mathcal{T}$, we let $E_0 = \mathcal{T}$ and, for $i > 0$,

$$E_i = \mathcal{E}(E_{i-1}) \cup \{C \sqsubseteq D \in \mathcal{T} \text{s.t. } \mathbf{T} \text{ does not occur in } C\}.$$

Observe that, being the knowledge base finite, there is an $n \geq 0$ such that, for all $m > n, E_m = E_n$ or $E_m = \emptyset$. The sequence $E_0 \supseteq E_1 \supseteq \ldots$ determines the rank (the exceptionality) of defeasible inclusions and of concepts in the RC wrt. the TBox $\mathcal{T}$. Concept C has *rank* i in RC (denoted $rank(C) = i$) iff i is the least natural number for which C is not exceptional for E_i. If C is exceptional for all E_i then $rank(C) = \infty$ (C has no rank). The rank of a typicality inclusion $\mathbf{T}(C) \sqsubseteq D$ is equal to $rank(C)$.

Example 2.4 *Let K be a knowledge base in Example 2.2 with TBox $\mathcal{T}$:*

$\mathbf{T}(Student) \sqsubseteq \neg Pay\_Taxes$
$\mathbf{T}(WStudent) \sqsubseteq Pay\_Taxes$
$\mathbf{T}(Student) \sqsubseteq Smart$
$WStudent \sqsubseteq Student$

It is possible to see that, from the construction above, we get:
$E_0 = \mathcal{T}$
$E_1 = \{\mathbf{T}(WStudent) \sqsubseteq Pay\_Taxes,\ WStudent \sqsubseteq Student\}$.
In particular, concept $Student$ has rank 0*, while $WStudent$ has rank* 1*. The rank of $Student$ is* 0*, as $Student$ is non-exceptional for E_0, i.e. $E_0 \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg Student$. In fact, there is a model $\mathcal{M}$ of E_0 (i.e., of $\mathcal{T}$) containing a domain element $x \in \Delta$ with rank* 0 *(an instance of $\mathbf{T}(\top)$), which is also an instance of $Student$*[2]*. Instead, $WStudent$ has rank* 1*, as it is exceptional for E_0: i.e. $E_0 \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg$*

[2]The instances of $\mathbf{T}(\top)$ in a model are all the elements of the domain which are maximally typical (as they have rank 0). Observe that here, as in Lehmann's semantics of the RC, we use a global notion of preference $<$, so that $\mathbf{T}(\top)$ can be intended as the set of domain elements which are maximally typical under "any" respect, in that they do not violate any defeasible inclusion of the knowledge base. A multipreference semantics, in which an element can be more typical than another under some respect but less typical under another one, was considered in [44], as a way of defining a refinement of the rational closure and cope with the inheritance blocking problem.

$WStudent$. In fact, no model of $\mathcal{T}$ can contain an element y with rank 0 which is an instance of $WStudent$ (as such a y would be a typical $WStudent$ and hence a tax payer, but also an typical $Student$ and, hence, would not be a tax payer, a contradiction).

One can see that the higher is the rank, the more exceptional (and specific) is a concept (e.g., $WStudent$ is more specific than $Student$; $WStudent \sqcap Italian \sqcap \neg Pay\_Taxes$ is more specific than $WStudent$). In particular, the rank of the concepts $Student \sqcap Italian$, and $Student \sqcap Italian \sqcap \neg Pay\_Taxes$ is 0; the rank of concepts $Student \sqcap Italian \sqcap Pay\_Taxes$, $WStudent \sqcap Italian$ and $WStudent \sqcap Italian \sqcap Pay\_Taxes$ is 1; and the rank of $WStudent \sqcap Italian \sqcap \neg Pay\_Taxes$ is 2.

Rational closure builds on the notion of exceptionality:

Definition 2.5 (Rational closure of TBox) *Let $K = (\mathcal{T}, \mathcal{A})$ be a DL knowledge base. The* rational closure of $\mathcal{T}$ *is defined as:*

$$RC(\mathcal{T}) = \{\mathbf{T}(C) \sqsubseteq D \mid \text{ either } rank(C) < rank(C \sqcap \neg D) \text{ or } rank(C) = \infty\} \cup \{C \sqsubseteq D \in \mathcal{T} \mid \mathcal{T} \models_{\mathcal{ALC}+\mathbf{T}_R} C \sqsubseteq D\}$$

where C and D are $\mathcal{ALC}$ concepts.

In Example 2.4, the query $\mathbf{T}(Student \sqcap Italian) \sqsubseteq \neg Pay\_Taxes$ is in the rational closure of the TBox $\mathcal{T}$, as $rank(Student \sqcap Italian) < rank(Student \sqcap Italian \sqcap Pay\_Taxes)$; so is the query $\mathbf{T}(WStudent \sqcap Italian) \sqsubseteq Pay\_Taxes$.

Exploiting the fact that entailment in $\mathcal{ALC} + \mathbf{T}_R$ can be polynomially encoded into entailment in $\mathcal{ALC}$, it is easy to see that deciding if an inclusion $\mathbf{T}(C) \sqsubseteq D$ belongs to the rational closure of TBox is a problem in EXPTIME and requires a polynomial number of entailment checks to an $\mathcal{ALC}$ knowledge base. In [40] it is also shown that the semantics corresponding to rational closure can be given in terms of *minimal canonical* $\mathcal{ALC}+\mathbf{T}_R$ models. In such models the rank of domain elements is minimized to make each domain element as typical as possible. Furthermore, canonical models are considered in which all possible combinations of concepts are represented. This is expressed by the following definitions.

Definition 2.6 (Minimal models of K) *Given two ranked models $\mathcal{M} = \langle \Delta, <, I \rangle$ and $\mathcal{M}' = \langle \Delta', <', I' \rangle$ of K, we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ ($\mathcal{M} \prec \mathcal{M}'$) if: $\Delta = \Delta'$; $C^I = C^{I'}$ for all (non-extended) concepts C; and, for all $x \in \Delta$, it holds that $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists $y \in \Delta$ such that $k_{\mathcal{M}}(y) < k_{\mathcal{M}'}(y)$.*

Given a knowledge base $K = (\mathcal{T}, \mathcal{A})$, we say that $\mathcal{M}$ is a minimal model of K (with respect to TBox) if it is a model satisfying K and there is no model $\mathcal{M}'$, satisfying K, such that $\mathcal{M}' \prec \mathcal{M}$.

The models corresponding to rational closure are required to be canonical. This property, expressed by the following definition, is needed when reasoning about the (relative) rank of the concepts: it is important to have them all represented by some instance in the model.

Definition 2.7 (Canonical model) *Given $K = (\mathcal{T}, \mathcal{A})$, a model $\mathcal{M} = \langle \Delta, <, I \rangle$ satisfying K is* canonical *if for each set of non-extended concepts $\{C_1, C_2, \ldots, C_n\}$ consistent with K (i.e., such that $K \not\models_{\mathcal{ALC}+\mathbf{T}_R} C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n \sqsubseteq \bot$), there exists (at least) a domain element $x \in \Delta$ such that $x \in (C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n)^I$.*

Definition 2.8 (Minimal canonical models of K) *$\mathcal{M}$ is a minimal canonical (ranked) model of K, if it is a ranked canonical model of K and it is minimal with respect $\prec$ (see Definition 2.6) among the ranked canonical models of K.*

The following result from [40] establishes a correspondence between satisfiability of a subsumption in minimal canonical models and the rational closure construction.

Theorem 2.9 *Let $K = (\mathcal{T}, \mathcal{A})$ be a knowledge base and $C \sqsubseteq D$ a query (with C an extended concept). $C \sqsubseteq D \in RC(\mathcal{T})$ if and only if $C \sqsubseteq D$ holds in all minimal canonical models of K.*

Furthermore, by Proposition 13 in [40]: the rank $k_{\mathcal{M}}(C)$ of a concept C in any minimal canonical model $\mathcal{M}$ of K coincides with the rank $rank(C)$ assigned to C by the rational closure construction, when $rank(C)$ is finite. When $rank(C) = \infty$, the concept C is not satisfiable in any model of the knowledge base.

Example 2.10 *Considering again the KB in Example 2.4, we can see that defeasible inclusions $\mathbf{T}\ (Student \sqcap\ Italian) \sqsubseteq \neg Pay\_Taxes$ and $\mathbf{T}(\ WStudent\ Italian) \sqsubseteq Pay\_Taxes$ are satisfied in all the minimal canonical models of K. In fact, for the first inclusion, in all the minimal canonical models of K, $Student \sqcap Italian$ has rank 0, while $Student \sqcap\ Italian \sqcap Pay\_Taxes$ has rank 1. Thus, in all the minimal canonical models of K each typical Italian student must be an instance of $\neg Pay\_Taxes$.*

Instead, the defeasible inclusion $\mathbf{T}(WStudent) \sqsubseteq Smart$ is not minimally entailed from K and, consistently, this inclusion does not belong to the rational closure of $\mathcal{T}$. Indeed, the concept $WStudent$ is exceptional for E_0, as it violates the defeasible property of students that, normally, do not pay taxes. For this reason, $WStudent$ does not inherit "any" of the defeasible properties of $Student$, the well known "blocking of property inheritance problem" of rational closure [56].

To overcome this weakness of RC, Lehmann introduced the notion of lexicographic closure [51], which strengthens the rational closure by allowing, roughly speaking, a class to inherit as many as possible of the defeasible properties of more general classes, giving preference to the more specific ones. The lexicographic closure has been extended to the description logic $\mathcal{ALC}$ by Casini and Straccia in [23]. In the example above, the property of students of being smart would be inherited by working students, as it is consistent with all other (strict or defeasible) properties of working students. In the general case, there may be exponentially many alternative bases (sets of defaults) to be considered for a given concept, which are all maximally preferred, and the lexicographic closure has to consider all of them to determine which defeasible inclusions can be accepted. The next section proposes a weaker approach, which leads to the construction of a single base for each concept.

3 The Skeptical Closure

Given an $\mathcal{ALC}$ concept B, one wants to identify the defeasible properties of typical B-elements (if any). Assume that the rational closure of the knowledge base K has already been constructed and k is the (finite) rank of concept B in the rational closure[3]. The typical B elements in the minimal canonical models of K are clearly compatible (by construction) with all the defeasible inclusions in E_k (by Definition 2.3 in the previous section) , but they might satisfy further defeasible inclusions with lower rank, i.e. those belonging to $E_0, E_1, \ldots, E_{k-1}$.

For instance, in the example above, concept $WStudent$ has rank 1, and for working students all the defeasible inclusions in set E_1 above hold (and, in particular, typical working students pay taxes). Among the defeasible inclusions in E_0, while the defeasible inclusion $\mathbf{T}(Student) \sqsubseteq \neg Pay\_Taxes$ is not compatible with the above property of typical students, the defeasible property $\mathbf{T}(Student) \sqsubseteq Smart$ is compatible, and there may be typical $WStudent$ which are Smart.

In general, there may be alternative maximal sets of defeasible inclusions compatible with B, among which one would prefer those that maximize the sets of defeasible inclusions with higher rank. This is indeed what is done by the lexicographic closure [51], which considers alternative maximally preferred sets of defaults called "bases", that, roughly speaking, maximize the number of defaults with higher ranks with respect to those with lower ranks (degree of seriousness), and where situations which violate a number of defaults with a certain rank are considered to be less plausible than situations which violate a lower number of defaults with the same rank. In general, there may be exponentially many alternative sets of defeasible inclusions (called bases in [51]) which are maximal and consistent for a given concept B, and the lexicographic closure has to consider all of them to determine if a defeasible inclusion is to be accepted or not. As a difference, in the following we define a construction which skeptically builds a single set of defeasible inclusions compatible with a concept B. The advantage of this construction is that it only requires (for each concept B) a polynomial number of calls to the underlying preferential $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ reasoner.

Let B be a concept with rank k in the rational closure. In order to see which are the defeasible inclusions compatible with B (beside those in E_k), we first single out the defeasible inclusions which are individually consistent with B and E_k. This is done while building the set S^B of the defeasible inclusions which are not overridden by those in E_k. As the set S^B might not be globally consistent with B, for the presence of conflicting defaults, we will consider the sets of defaults in S^B with the same rank, going from $k - 1$ to 0 and we will add them to E_k, if consistent (starting from the highest rank). When we find an inconsistency among defaults with rank i in S^B, we stop. In this way, we extend E_k with all defeasible inclusions in S^B with rank from $k - 1$ to $i + 1$, which are not conflicting with each other and can be inherited by B instances (even though the construction of rational closure has excluded them from E_k). Instead, as there is some conflict among defaults with rank i in S^B, we exclude all defaults with rank from 0 to i.

[3]When $rank(B) = \infty$, the defeasible inclusion $\mathbf{T}(B) \sqsubseteq D$ belongs to the rational closure of TBox for any D. Hence, we assume $\mathbf{T}(B) \sqsubseteq D$ also belongs to the skeptical closure, and we defer considering this case until Definition 3.5. So far, we always assume k to be finite.

For an $\mathcal{ALC}$ concept B with $rank(B) = k$, let us define the set S^B of typicality inclusions $\mathbf{T}(C) \sqsubseteq D$ in the TBox $\mathcal{T}$ which are *individually compatible with* B *with respect to* E_k as:

$$S^B = \{\mathbf{T}(C) \sqsubseteq D \in \mathcal{T} \mid \ E_k \cup \{\mathbf{T}(C) \sqsubseteq D\} \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg B\}^4.$$

For instance, in Example 2.4, for $B = WStudent$, which has rank 1, we have that

$$S^{WStudent} = \{\mathbf{T}(Student) \sqsubseteq Smart,\ \mathbf{T}(WStudent) \sqsubseteq PayTaxes\}$$

is the set of defeasible inclusions compatible with $WStudent$ wrt. E_1. The defeasible inclusion $\mathbf{T}(Student) \sqsubseteq \neg Pay\_Taxes$ is not included in $S^{WStudent}$ as it is not (individually) compatible with $WStudent$ (the conflicting default $\mathbf{T}(WStudent) \sqsubseteq PayTaxes$ with rank 1 overrides it).

Clearly, although each defeasible inclusion in S^B is compatible with B, it might be the case that overall the set S^B is not compatible with B, i.e., $E_k \cup S^B \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}}$ $\mathbf{T}(\top) \sqsubseteq \neg B$.

Let us consider the following variant of Example 2.4.

Example 3.1 *Let* $\mathcal{T}$ *be the TBox:*
$\mathbf{T}(Student) \sqsubseteq Young$
$\mathbf{T}(Student) \sqsubseteq \neg PayTaxes$
$\mathbf{T}(Employee) \sqsubseteq PayTaxes$
$\mathbf{T}(Student \sqcap Employee) \sqsubseteq \neg Young$
Let $B = Student \sqcap Employee$. *While concepts Student and Employee have rank* 0, *the concept Student* $\sqcap$*Employee has rank* 1. *In this example,* $E_0 = \mathcal{T}$ *and*

$$E_1 = Strict_{\mathcal{T}} \cup \{\mathbf{T}(Student \sqcap Employee) \sqsubseteq \neg Young\}.$$

The property that typical employed students are not young overrides the property that students are typically young. Indeed the default $\mathbf{T}(Student) \sqsubseteq Young$ *is not individually compatible with Student*$\sqcap$ *Employee. Instead, the defeasible properties* $\mathbf{T}(Student) \sqsubseteq \neg PayTaxes$ *and* $\mathbf{T}(Employee) \sqsubseteq Pay$- *Taxes are both individually compatible with Student*$\sqcap$ *Employee, and*

$$S^B = \{\mathbf{T}(Student) \sqsubseteq \neg PayTaxes,\ \mathbf{T}(Employee) \sqsubseteq PayTaxes\}.$$

Nevertheless, the overall set S^B *is not compatible with Student* $\sqcap$ *Employee. In fact, the two defeasible inclusions in* S^B *are conflicting.*

When compatible with B, S^B is the unique maximal basis with respect to the *seriousness ordering* in [51] (as defined for constructing the lexicographic closure). However, when S^B is not compatible with B, we cannot use all the defeasible inclusions in S^B to derive conclusions about typical B elements. In this case, we could either use the defeasible inclusions in E_k only, as in the rational closure, or we could additionally

[4]Notice that the defeasible inclusions with rank $\geq k$ are already in E_k and are always compatible with E_k. For each B, the rank $rank(B) = k$ is used here and below to select the set E_k with respect to which compatibility is to be checked.

use a subset of the defeasible inclusions S^B with rank less than $k = rank(B)$ (i.e., inclusions not in E_k). This is essentially what is done in the lexicoghaphic closure, where (in essence) the most preferred subsets of S^B are selected according to a lexicographic order, which prefers defaults with higher ranks to defaults with lower ranks. In our construction, instead, we consider the subsets $S^B_{k-1}, \ldots, S^B_1, S^B_0$ of the set S^B defined above, adding to E_k all the defeasible inclusions in S^B with rank $k-1$ (let this set be S^B_{k-1}), provided they are (altogether) compatible with B wrt. E_k. Then, we add all the defeasible inclusions in S^B with rank $k-2$ which are individually compatible with B wrt. $E_k \cup S^B_{k-1}$ (let this set be S^B_{k-2}), provided they are altogether compatible with B wrt. $E_k \cup S^B_{k-1}$, and so on and so forth, for lower ranks, until a set S^B_{h-1} is found, which is incompatible with the previous inclusions and E_k. This leads to the construction below.

Definition 3.2 *Let $K = (\mathcal{T}, \mathcal{A})$ be a knowledge base and B a concept such that $rank(B) = k$. Given two sets of defeasible inclusions $S, S' \subseteq \mathcal{T}$, S* is globally compatible with *B* w.r.t. *$E_k \cup S'$ if*

$$E_k \cup S \cup S' \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg B$$

Definition 3.3 *Let K be a knowledge base and B a concept such that $rank(B) = k$. The* skeptical closure of *K* wrt. *B is the set of inclusions $S^{sk,B} = E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S^B_h$ where:*

- *$S^B_i \subseteq E_i - E_{i+1}$ is the set of defeasible inclusions with rank i which are* individually compatible with *B* wrt. *$E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S^B_{i+1}$ (for each finite rank $i < k$);*
- *h is the least j (for $0 \leq j < k$) such that S^B_j is* globally compatible with *B* wrt. *$E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S^B_{j+1}$, if such a j exists; $S^{sk,B} = E_k$, otherwise.*

Intuitively, $S^{sk,B}$ contains, for each rank j, all the defeasible inclusions having rank j which are compatible with B and with the more specific defeasible inclusions (having rank $> j$). As S^B_{h-1} is not included in the skeptical closure, it must be that $E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S_h \cup S^B_{h-1} \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg B$, i.e., the set S^B_{h-1} contains conflicting defeasible inclusions which are not overridden by more specific ones. In this case, the inclusions in S^B_{h-1} (and, similarly, all the defeasible inclusions with rank lower than $h-1$) are not included in the skeptical closure w.r.t. B.

Example 3.4 *For the knowledge base K in Example 2.4, where $B = WStudent$ has rank 1, we have $S^B_0 = \{\mathbf{T}(Student) \sqsubseteq Smart\}$, which is compatible with $WStudent$ wrt. E_1. Hence, $S^{sk,B} = E_1 \cup S^B_0$.*

Let us define when a defeasible inclusion belongs to the skeptical closure of a TBox $\mathcal{T}$.

Definition 3.5 *Let $K = (\mathcal{T}, \mathcal{A})$ be a knowledge base and $\mathbf{T}(B) \sqsubseteq D$ a query. $\mathbf{T}(B) \sqsubseteq D$* is in the skeptical closure of *$\mathcal{T}$ if either $rank(B) = \infty$ holds in the rational closure construction of $\mathcal{T}$, or $rank(B)$ is finite and $S^{sk,B} \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg B \sqcup D)$ (for B and D $\mathcal{ALC}$ concepts).*

Once the rational closure of $\mathcal{T}$ has been computed, the identification (for a given concept B) of the defeasible inclusions in $S^{sk,B}$ requires a number of entailment checks which is linear in the number of defeasible inclusions in TBox. First, the individual compatibility with B of each defeasible inclusion in $\mathcal{T}$ (wrt. some set $E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S^B_{i+1}$, which depends on the rank i of the inclusion) has to be checked to compute all the S^B_i's (requiring one $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ entailment check, for each defeasible inclusion). Then, a compatibility check is needed, to verify the global compatibility of S^B_i, for each rank i from $k-1$ to 0, in the worst case. As the maximum number or ranks in the rational closure is bounded by the number of defeasible inclusions in TBox (but it might be significantly lower in practical cases), computing the skeptical closure for a concept B requires a number of entailment checks which is, in the worst case, $O(2 \times |\mathcal{T}|)$. As deciding $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ entailment is in EXPTIME [42, 36], checking whether a query $\mathbf{T}(C) \sqsubseteq D$ is in the skeptical closure of a TBox $\mathcal{T}$ is still a problem in EXPTIME.

Although computing the rational closure is already EXPTIME-hard (from hardness of subsumption in $\mathcal{ALC}$ with general TBox [2]) differently from the RC construction, which requires a quadratic number of calls to an $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ reasoner to compute the TBox ranking, here, after computing the ranking, we still need a linear number of entailment checks for any concept B in a query $\mathbf{T}(B) \sqsubseteq D$.

Example 3.6 *For the knowledge base K in Example 2.4, we have seen that, for $B = WStudent$ (with rank 1), $S^B_0 = \{\mathbf{T}(Student) \sqsubseteq Smart\}$ is (globally) compatible with WStudent w.r.t. E_1, and $S^{sk,B} = E_1 \cup S^B_0$. It is possible to see that $S^{sk,B} \models_{\mathcal{ALC}+\mathbf{T}_R}$ $\mathbf{T}(\top) \sqsubseteq (\neg WStudent \sqcup Smart)$ (that is, for all the most typical elements of the domain satisfying $S^{sk,B}$, they are not WStudent's or they are Smart). $\mathbf{T}(WStudent) \sqsubseteq Smart$ is then in the skeptical closure of TBox. In this case, the typical property of students of being Smart is inherited by working students.*

Example 3.7 *For the knowledge base K' in Example 3.1, as we have seen, $B = Student \sqcap Employee$ has rank 1, $E_1 = \{\mathbf{T}(Student \sqcap Employee) \sqsubseteq \neg\ Young\}$, and $S^B = \{\mathbf{T}(Student) \sqsubseteq \neg PayTaxes,\ \mathbf{T}(Employee) \sqsubseteq PayTaxes\}$. In this case, as $S^B_0 = S^B$ contains conflicting defaults about tax payment, S^B_0 is not (globally) compatible with $Student \sqcap Employee$ and E_1, so that $S^{sk,B} = E_1$.*

Let us consider the following knowledge base from [34] to see that, in the skeptical closure, inheritance of defeasible properties, when not overridden by more specific concepts, applies to concepts of all ranks.

Example 3.8 *Consider a knowledge base $K = (\mathcal{T}, \mathcal{A})$, where $\mathcal{A} = \emptyset$ and $\mathcal{T}$ contains the following inclusions:*

$Penguin \sqsubseteq Bird$ $\quad$ $BabyPenguin \sqsubseteq Penguin$
$\mathbf{T}(Bird) \sqsubseteq Fly$ $\quad$ $\mathbf{T}(Bird) \sqsubseteq NiceFeather$
$\mathbf{T}(Penguin) \sqsubseteq \neg Fly$ $\quad$ $\mathbf{T}(Penguin) \sqsubseteq BlackFeather$
$\mathbf{T}(BabyPenguin) \sqsubseteq \neg BlackFeather$.

Here, we expect that the defeasible property of birds having a nice feather is inherited by typical penguins, even though penguins are exceptional birds regarding flying. We *also expect that typical baby penguins inherit the defeasible property of penguins that they do not fly, although the defeasible property* $BlackFeather$ *is instead overridden for typical baby penguins, and that they inherit the typical property of birds of having nice feather. We have that* $rank(Bird) = 0$, $rank(Penguin) = 1$, $rank(BabyPenguin) = 2$ *as, in the rational closure construction, inclusions* $\mathbf{T}(Bird) \sqsubseteq Fly$ *and* $\mathbf{T}(Bird) \sqsubseteq NiceFeather$ *have rank 0, while* $\mathbf{T}(Penguin) \sqsubseteq \neg Fly$ *and* $\mathbf{T}(Penguin) \sqsubseteq BlackFeather$ *have rank 1 and* $E_2 = Strict_{\mathcal{T}} \cup \{\mathbf{T}(BabyPenguin) \sqsubseteq \neg BlackFeather\}$.

For $B = BabyPenguin$*, we get* $S_1^B = \{\mathbf{T}(Penguin) \sqsubseteq \neg Fly\}$ *and* $S_0^B = \{\mathbf{T}(Bird) \sqsubseteq NiceFeather\}$*. Also,* S_1^B *is globally consistent with B wrt.* E_2*, and* S_0^B *is globally consistent with B wrt.* $E_2 \cup S_1^B$*. Hence,* $S^{sk,B} = E_2 \cup S_1^B \cup S_0^B = \{\mathbf{T}(BabyPenguin) \sqsubseteq \neg BlackFeather, \mathbf{T}(Penguin) \sqsubseteq \neg Fly, \mathbf{T}(Bird) \sqsubseteq NiceFeather\}$*. The query* $\mathbf{T}(BabyPenguin) \sqsubseteq NiceFeather \sqcap \neg Fly \sqcap \neg BlackFeather$ *is in the skeptical closure of TBox* $\mathcal{T}$*, as* $S^{sk,B} \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg BabyPenguin \sqcup (NiceFeather \sqcap \neg Fly \sqcap \neg BlackFeather)$.

To see that the notion of skeptical closure is rather weak, let us slightly modify Example 3.1.

Example 3.9 *Let us remove the last inclusion from the TBox* $\mathcal{T}$ *in Example 3.1:*

$\mathbf{T}(Student) \sqsubseteq Young$
$\mathbf{T}(Student) \sqsubseteq \neg PayTaxes$
$\mathbf{T}(Employee) \sqsubseteq PayTaxes$

Let $B = Student \sqcap Employee$*. As in Example 3.1, the rational closure assigns rank* 0 *to concepts Student and Employee and rank* 1 *to* $Student \sqcap Employee$*. In this case,* $E_0 = \mathcal{T}$*,* $E_1 = \emptyset$ *and*

$$S_0^B = \{\mathbf{T}(Student) \sqsubseteq \neg Pay\_Taxes,\ \mathbf{T}(Student) \sqsubseteq Young,\ \mathbf{T}(Employee) \sqsubseteq PayTaxes\}.$$

As S_0^B *is not (globally) compatible with* $Student \sqcap Employee$ *and* E_1*, again* $S^{sk,B} = E_1$*. Therefore, the defeasible property that typical students are young is not inherited by typical employed students, and the inclusion* $\mathbf{T}(Student \sqcap Employee) \sqsubseteq Young$ *is not in the skeptical closure of* $\mathcal{T}$.

The skeptical closure is a weak construction: in Example 3.9 due to the conflicting defaults concerning tax payment for $Employee$ and $Student$ (both having rank 0) also the property that typical students are young is not inherited by the typical employed students. Notice that, the property that typical employed students are young would be accepted in the lexicographic closure of K', as there are two bases, the one including $\mathbf{T}(Student) \sqsubseteq \neg Pay\_Taxes$ and the other one including $\mathbf{T}(Employee) \sqsubseteq Pay\_Taxes$, both containing $\mathbf{T}(Student) \sqsubseteq Young$.

In the next section, we introduce a semantics based on two preference relations. We will show that this semantics characterizes a variant of the lexicographic closure introduced in [34] and exploit it to define a semantic construction for the weaker skeptical closure.

4 Refined, bi-preference interpretations

To capture the semantics of the skeptical closure, we build on the preferential semantics for rational closure of $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$, introducing a notion of *refined, bi-preference interpretation* (for short, BP-interpretation), which contains an additional notion of preference with respect to an $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ interpretation. We let an interpretation to be a tuple $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$, where the triple $\langle \Delta, <_{rc}, I \rangle$ is a ranked interpretation as defined in Section 2 and $<$ is an additional preference relation over Δ, with the properties of being irreflexive, transitive and well-founded (but we do not require modularity of $<$). In BP-interpretations, $<$ represents a refinement of $<_{rc}$.

Definition 4.1 (BP-interpretation) *Given a knowledge base K, a* bi-preference interpretation *(or BP-interpretation) is a structure $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$, where Δ is a domain, I is an interpretation function as defined in Definition 2.1, where, in particular, $(\mathbf{T}(C))^I = min_<(C^I)$, and $<_{rc}$ and $<$ are preference relations over Δ, with the properties of being irreflexive, transitive, well-founded. Furthermore $<_{rc}$ is modular.*

The bi-preference semantics builds on a ranked semantics for the preference relation $<_{rc}$, providing a characterization of the rational closure of K, and exploits it to define the preference relation $<$ which is not required to be modular. As we will see, this semantics provides a sound and complete characterization of a variant of the lexicographic closure for $\mathcal{ALC}$, the multipreference closure (MP-closure, for short), first introduced in [34], and we will use it to define a semantic characterization of the skeptical closure. The BP-semantics is weaker than the multipreference semantics in [44] (as the MP-closure is a sound but incomplete construction for the multipreference semantics). It does not exploit multiple preferences w.r.t. aspects and it directly builds on the preference relation $<_{rc}$.

Let $k_{\mathcal{M},rc}$ be the ranking function associated in $\mathcal{M}$ with the modular relation $<_{rc}$, which is defined as the ranking function $k_{\mathcal{M}}$ for ranked models in Section 2. Similarly, the ranking function is extended to concepts by letting the rank $k_{\mathcal{M},rc}(C)$ of a concept C in a BP-interpretation $\mathcal{M}$ (w.r.t. the preference relation $<_{rc}$) to be $k_{\mathcal{M},rc}(C) = min\{k_{\mathcal{M},rc}(x) : x \in C^I\}$.

Given a BP-interpretation $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ and an element $x \in \Delta$, we say that x *violates the typicality inclusion* $\mathbf{T}(C) \sqsubseteq D$ if $x \in (C \sqcap \neg D)^I$, and that x *satisfies* $\mathbf{T}(C) \sqsubseteq D$ if $x \notin (C \sqcap \neg D)^I$. Let us define when a BP-interpretation is a model of a knowledge base K:

Definition 4.2 (BP-model of K) *Given a knowledge base $K = (\mathcal{T}, \mathcal{A})$, a BP-interpretation $\mathcal{M} = \langle \Delta, <_{rc} <, I \rangle$ is a* BP-model of K *if it satisfies both its TBox $\mathcal{T}$ and its ABox $\mathcal{A}$, in the following sense:*

(1) for all strict inclusions $C \sqsubseteq D$ in $\mathcal{T}$ (i.e., $\mathbf{T}$ does not occur in C), $C^I \subseteq D^I$;

(2) for all typicality inclusions $\mathbf{T}(C) \sqsubseteq D$ in $\mathcal{T}$, $min_{<_{rc}}(C^I) \subseteq D^I$;

(3) $<$ satisfies the following specificity condition:

$x < y$ if (i) y violates some defeasible inclusion $\mathbf{T}(C) \sqsubseteq D \in \mathcal{T}$ satisfied by x and
(ii) for every $\mathbf{T}(C_j) \sqsubseteq D_j \in \mathcal{T}$, which is violated by x and satisfied by y,
there is a $\mathbf{T}(C_k) \sqsubseteq D_k \in \mathcal{T}$, which is violated by y and satisfied by x,
such that $k_{\mathcal{M},rc}(C_j) < k_{\mathcal{M},rc}(C_k)$.

(4) for all $C(a)$ in ABox, $a^I \in C^I$; and, for all $R(a, b)$ in ABox, $(a^I, b^I) \in R^I$;

While the satisfiability conditions (1), (2) and (4) are the same as in Section 2 for the ranked model $\langle \Delta, <_{rc}, I \rangle$, the specificity condition (3) requires the relation $<$ to satisfy the condition that, if y violates defeasible inclusions more specific than those violated by x, then $x < y$ (in particular, the condition $k_{\mathcal{M},rc}(C_j) < k_{\mathcal{M},rc}(C_k)$ means that concept C_k is more specific than concept C_j, as C_k has a higher rank in the rational closure).

In the definition above we do not impose the further requirement that, for all inclusions $\mathbf{T}(C) \sqsubseteq D$, $min_<(C^I) \subseteq D^I$ holds. However, we can easily see that this condition follows from condition (2) and from the property that $<_{rc} \subseteq <$ holds.

Proposition 4.3 *Given a knowledge base K and a BP-model $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ of K, $<_{rc} \subseteq <$.*

Proof 4.4 *We show that $x <_{rc} y$ implies $x < y$. If $x <_{rc} y$, then for some r, $k_{\mathcal{M},rc}(x) = r < k_{\mathcal{M},rc}(y)$. As $\mathcal{M}$ is a minimal canonical BP-model of K, by the correspondence with the rational closure, x satisfies all the defeasible inclusions in E_r. Instead, y falsifies some defeasible inclusion $\mathbf{T}(C_k) \sqsubseteq D_k$ with $rank(C_k) = r$. As x can only falsify defeasible inclusions with rank less then r, by condition (3) in Definition 4.2, $x < y$. Therefore, $<_{rc} \subseteq <$.*

Corollary 4.5 *Given a knowledge base K and a BP-model $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ of K, for all inclusions $\mathbf{T}(C) \sqsubseteq D$, $min_<(C^I) \subseteq D^I$ holds.*

Proof 4.6 *From item (2) in Definition 4.2, we know that $min_{<_{rc}}(C^I) \subseteq D^I$. By Proposition 4.3, $<_{rc} \subseteq <$, from which it follows that $min_<(C^I) \subseteq min_{<_{rc}}(C^I)$. Hence, the thesis follows.*

We define logical entailment under the BP-semantics as follows: a query F (with form $C_L(a)$ or $C_L \sqsubseteq C_R$) is logically entailed by K in the BP-semantics (written $K \models_{BP} F$) if F holds in all BP-models of K. The following result can be easily proved for BP-entailment:

Theorem 4.7 *(a) If $K \models_{\mathcal{ALC}+\mathbf{T}_R} F$ then also $K \models_{BP} F$; (b) if $\mathbf{T}$ does not occur in F the other direction also holds: If $K \models_{BP} F$ then also $K \models_{\mathcal{ALC}+\mathbf{T}_R} F$.*

Proof 4.8 *(Sketch) For part (a), by contraposition, assume $K \not\models_{BP} F$. Let $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ be a BP-model of K falsifying F. It is easy to see that $\mathcal{N} = \langle \Delta, <, I \rangle$ is a preferential model of K (which follows from (1) and (4) in Definition 4.2 and from*

Corollary 4.5), and that $\mathcal{N}$ falsifies F. From $\mathcal{N}$ we can build a ranked model $\mathcal{N}'$ of K falsifying F by letting, for all $x \in \Delta$, $k_{\mathcal{N}'}(x)$ to be the length of the longest chain $w_0 < \cdots < w$ from w to a minimal w_0 (i.e., there is no w' such that $w' < w_0$). Therefore, $K \not\models_{\mathcal{ALC}+\mathbf{T}_R} F$.

For part (b), by contraposition, if $K \not\models_{\mathcal{ALC}+\mathbf{T}_R} F$, then there is an $\mathcal{ALC} + \mathbf{T}_R$ model $\mathcal{N} = \langle \Delta, <_{rc}, I \rangle$ of K falsifying F. We can define a BP-model of K, $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$, letting $x < y$ iff (i) and (ii) in Definition 4.2 hold. It is easy to see that $<$ is irreflexive, transitive and well-founded and, by construction, $\mathcal{M}$ satisfies conditions (1)-(4) of Definition 4.2. Hence, $\mathcal{M}$ is an BP-model of K. Also, as F does not contain typicality inclusions, it is easy to see that $\mathcal{M}$ falsifies F, i.e., $K \not\models_{BP} F$.

To define a notion of minimal canonical BP-model for K, we proceed as in the semantic characterization of the rational closure in Section 2. Let $d_{\mathcal{M}}$ be a function associated with the preference relation $<$ such that, for any element $x \in \Delta$: if $x \in min_{<}(\Delta)$, then $d_{\mathcal{M}}(x) = 0$; otherwise, $d_{\mathcal{M}}(x)$ is the length of the longest path $x_0 < x_1 < \ldots < x$ from x to an element x_0 such that $d_{\mathcal{M}}(x) = 0$.

Although $<$ is not assumed to be modular, for each domain element x, $d_{\mathcal{M}}(x)$ represents the distance of x from the most preferred elements in the model, and can be used for defining a notion of preference $\prec_{BP}$ among BP-models of K. Let $Min_{RC}(K)$ be the set of all BP-models $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ of K such that $\langle \Delta, <_{rc}, I \rangle$ is a minimal canonical model of K according to the semantics of rational closure in Section 2 (Definition 2.8). Thus, the models in $Min_{RC}(K)$ are those built from the minimal canonical models of the rational closure of K. The minimal (canonical) BP-models of K will be the models in $Min_{RC}(K)$ which also minimize the distance $d_{\mathcal{M}}(x)$ of each domain element x.

Definition 4.9 (Minimal canonical BP-Models) *Given two BP-models of K, $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ and $\mathcal{M}' = \langle \Delta', <'_{rc}, <', I' \rangle$ in $Min_{RC}(K)$,* $\mathcal{M}'$ is preferred to $\mathcal{M}$ *(written $\mathcal{M}' \prec_{BP} \mathcal{M}$) if*

- *$\Delta = \Delta'$, $I = I'$, and*
- *for all $x \in \Delta$, $d_{\mathcal{M}'}(x) \leq d_{\mathcal{M}}(x)$;*
- *for some $y \in \Delta$, $d_{\mathcal{M}'}(y) < d_{\mathcal{M}}(y)$*

A BP-interpretation $\mathcal{M}$ is a minimal canonical BP-model *of K if $\mathcal{M}$ is a model of K, $\mathcal{M} \in Min_{RC}(K)$ and there is no $\mathcal{M}' \in Min_{RC}(K)$ such that $\mathcal{M}' \prec_{BP} \mathcal{M}$.*

We denote by $\models^{min}_{BP}$ the entailment with respect to minimal canonical BP-models: for a query F, $K \models^{min}_{BP} F$ if F is satisfied in all the minimal canonical BP-models of K.

Observe that, according to this definition, for computing the minimal (canonical) BP-models of K one first needs to compute the set of the minimal (canonical) models of K which characterize rational closure of K. Then, among such models, one can select those which are minimal with respect to $\prec_{BP}$.

Clearly, as for minimal canonical BP-models $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ of a KB, $\langle \Delta, <_{rc}, I \rangle$ is also a minimal ranked model of the RC, as defined in Section 2, $<_{rc}$ corresponds to the preference relation in minimal canonical models of the rational closure of $\mathcal{ALC}$, and the rank $k_{\mathcal{M},rc}(x)$ of domain elements will be the same as in the minimal models of rational closure. Thus, by Theorem 2.9, the value of $k_{\mathcal{M},rc}(C)$, for any concept C, in a minimal canonical BP-model is equal to $rank(C)$, the rank assigned to C by the rational closure construction in Section 2. The rank of domain elements with respect to $<_{rc}$ is used to determine the preference relation $<$ on domain elements, according to condition (3).

Example 4.10 *Let us consider the TBox $\mathcal{T}$ in Example 2.4:*

$\mathbf{T}(Student) \sqsubseteq \neg Pay\_Taxes$
$\mathbf{T}(WStudent) \sqsubseteq Pay\_Taxes$
$\mathbf{T}(Student) \sqsubseteq Smart$
$WStudent \sqsubseteq Student$

In all minimal canonical BP-models $\mathcal{M}$, $k_{\mathcal{M},rc}(Student) = 0$, while $k_{\mathcal{M},rc}(WStudent)$ $= k_{\mathcal{M},rc}$ $(WStudent \sqcap Smart) = k_{\mathcal{M},rc}(WStudent \sqcap \neg Smart) = 1$, as in the model of the rational closure. Let x and y be two elements in the domain of $\mathcal{M}$ such that: $k_{\mathcal{M},rc}(x) = k_{\mathcal{M},rc}(y) = 1$, $x \in WStudent \sqcap Pay\_Taxes \sqcap Smart$, and $y \in$ WStudent *$\sqcap Pay\_Taxes \sqcap \neg Smart$. Such elements x and y exist in $\mathcal{M}$, as $\mathcal{M}$ is canonical, and the two concepts $WStudent \sqcap Pay\_Taxes \sqcap Smart$, and $WStudent \sqcap Pay\_Taxes$ $\sqcap \neg Smart$ have rank 1 in the RC. As y violates the typicality inclusion $\mathbf{T}(Student) \sqsubseteq$ $Smart$, which is satisfied by x, and there is no typicality inclusion which is satisfied by y and violated by x, by condition (3) in Definition 4.2, it must be that $x < y$.*

Hence, in all the minimal canonical BP-models $\mathcal{M}$ of the KB, the domain elements z which are instances of $\mathbf{T}(WStudent)$, not only must be instances of $WStudent \sqcap$ Pay_Taxes (as the defeasible inclusion $\mathbf{T}(WStudent) \sqsubseteq Pay\_Taxes$ must be satisfied by all typical working students), but also must be instances of $WStudent \sqcap Pay\_Taxes$ $\sqcap Smart$, as they are preferred in $\mathcal{M}$ to $WStudent \sqcap Pay\_Taxes \sqcap \neg Smart$ instances. Therefore, $\mathbf{T}(WStudent) \sqsubseteq Smart$ holds in $\mathcal{M}$.

In Example 2.4 entailment in minimal canonical BP-models captures the defeasible inclusions which belong to the skeptical closure. However, this is not the general case.

Example 4.11 *Let us consider, as a variant of Example 3.1, a knowledge base $K =$ $(\mathcal{T}, \mathcal{A})$ with $\mathcal{A} = \emptyset$ and the following TBox $\mathcal{T}$:*

$\mathbf{T}(Student) \sqsubseteq Young$
$\mathbf{T}(Student) \sqsubseteq \neg PayTaxes \sqcap \exists hasSSN.\top$
$\mathbf{T}(Employee) \sqsubseteq PayTaxes \sqcap \exists hasSSN.\top$
$\mathbf{T}(Student \sqcap Employee) \sqsubseteq \neg Young$

stating that typical students (and typical employee) have a social security number. As in Example 3.1 in all the minimal canonical BP-model $\mathcal{M}$ of K, we have $k_{\mathcal{M},rc}(Student)$= $k_{\mathcal{M},rc}$ $(Employee) = 0$ and $k_{\mathcal{M},rc}(Student \sqcap Employee) = 1$, as in the rational closure. As $E_1 = Strict_{\mathcal{T}} \cup \{\mathbf{T}(Student \sqcap Employee) \sqsubseteq \neg Young\}$, in the skeptical closure construction:

$$S_0^B = \{\mathbf{T}(Student) \sqsubseteq \neg PayTaxes \sqcap \exists hasSSN.\top, \mathbf{T}(Employee) \sqsubseteq PayTaxes \sqcap \exists hasSSN.\top\}$$

and the set S_0^B is not (globally) compatible with $Student \sqcap Employee$ and E_1, so that $S^{sk,B} = E_1$. Hence, $\mathbf{T}(Student \sqcap Employee) \sqsubseteq \exists hasSSN.\top$ is not in the skeptical closure of the KB. However, it is easy to see that this defeasible inclusion is satisfied in all the minimal canonical BP-models $\mathcal{M}$ of K. i.e., $K \models_{BP}^{min} \mathbf{T}(Student \sqcap Employee) \sqsubseteq \exists hasSSN.\top$.

To see why $K \models_{BP}^{min} \mathbf{T}(Student \sqcap Employee) \sqsubseteq \exists hasSSN.\top$, let $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ be a minimal canonical BP-model of K and let

$$y \in \mathbf{T}((Student \sqcap Employee))^I = min_<(Student \sqcap Employee)^I \subseteq min_{<_{rc}}(Student \sqcap Employee)^I$$

(the last inclusion follows from Proposition 4.3. To show that $y \in (\exists hasSSN.\top)^I$ suppose, for a contradiction, that $y \notin (\exists hasSSN.\top)^I$. As $y \in (Student \sqcap Employee)^I$, y violates both the second and the third defeasible inclusions in $\mathcal{T}$. In the canonical model $\mathcal{M}$, however, there must be an element $x \in min_{<_{rc}}(Student \sqcap Employee)^I$ such that $x \in (\neg PayTaxes \sqcap \exists hasSSN.\top)^I$, so that x does not violate the second defeasible inclusion $\mathbf{T}(Student) \sqsubseteq \neg PayTaxes \sqcap \exists hasSSN.\top$, which is violated by y. Also, x satisfies the inclusions in E_1, so that there is no inclusion which is violated by x and not by y. Hence, $x < y$ must hold in $\mathcal{M}$, by condition (3) of Definition 4.2, which contradicts the hypothesis that $y \in min_<(Student \sqcap Employee)^I$.

The example above shows that entailment in minimal canonical BP-models is too strong for providing a characterization of the skeptical closure: $\mathbf{T}(Student \sqcap Employee) \sqsubseteq \exists hasSSN.\top$ is minimally entailed by K, but it is not in the skeptical closure of K. In the next section we consider a stronger closure construction, which is characterized by minimal canonical BP-models and, from this result, in Section 6 we can provide a semantic construction for the skeptical closure.

The previous example clarifies that the skeptical closure, is syntax dependent and, in particular, a defeasible inclusion $\mathbf{T}(A) \sqsubseteq B \sqcap C$, in general, is not equivalent to the conjunction of $\mathbf{T}(A) \sqsubseteq B$ and $\mathbf{T}(A) \sqsubseteq C$. This is a problem of all refinements of the rational closure. As observed by Lehmann, the lexicographic closure construction is "extremely sensitive to the way defaults are presented" and "the way defaults are presented is important" [51]. The rational closure, instead, is not syntax dependent.

5 Correspondence between BP-models and a variant of lexicographic closure

In this section we show that the semantics of minimal canonical BP-models introduced in the previous section provides a characterization of the multipreference closure (MP-closure) for $\mathcal{ALC}$, a closure introduced in [34, 32] as a sound approximation of the multipreference semantics for $\mathcal{ALC}$ [44].

A study of the MP-closure in the propositional case has been done in [35], showing that the MP-closure can be regarded as a natural variant of Lehmann's lexicographic closure, when abandoning the maximal entropy assumption. In the propositional case, MP-closure is proved to be weaker than the lexicographic closure, but stronger than the relevant closure [17]. In the following, we consider the MP-closure construction for the description logic $\mathcal{ALC}$ and prove that the typicality inclusions which belong to the MP-closure of a TBox $\mathcal{T}$ are those entailed by $\mathcal{T}$ under the minimal canonical BP-models semantics defined in Section 4, thus providing a sound and complete characterization of the MP-closure for $\mathcal{ALC}$, that we will exploit in defining a characterization of the skeptical closure.

Let B be a concept with rank k. Informally, we want to consider all the possible maximal sets of typicality inclusions S which are compatible with E_k and with B, i.e. the maximal sets of defeasible properties that a B element can enjoy besides those in E_k. For instance, in Example 3.1, if $B = \mathit{Student} \sqcap \mathit{Employee}$, with $\mathit{rank}(B) = 1$, we have two possible alternative ways of maximally extending the set E_1, containing the defeasible inclusion $\mathbf{T}(\mathit{Student} \sqcap \mathit{Employee}) \sqsubseteq \neg \mathit{Young}$: either with the defeasible inclusion $\mathbf{T}(\mathit{Student}) \sqsubseteq \neg \mathit{Pay\_Taxes}$ or with the defeasible inclusion $\mathbf{T}(\mathit{Employee}) \sqsubseteq \mathit{Pay\_Taxes}$. As we have seen in Example 3.1, these two defeasible inclusions are conflicting, and in the skeptical closure we do not accept any of them. However, here we consider all alternative maximally consistent scenarios, compatible with the fact that the concept $B = \mathit{Student} \sqcap \mathit{Employee}$ is nonempty. In none of these scenarios the defeasible property that normally students are young can be accepted, as it is incompatible with the more specific property that normally students who are employee are not young.

Let $D_i = E_i - E_{i+1}$ be the set of typicality inclusions with rank $i \geq 0$. Given a set S of typicality inclusions of the TBox, we let $S_i = S \cap D_i$, for all ranks $i = 0, \ldots, n$ in the rational closure, thus defining a partition of the typicality inclusions with finite rank in S, according to their rank[5]. We introduce a preference relation among sets of typicality inclusions as follows:

Definition 5.1 *Let $S, S' \subseteq \mathcal{T} \backslash \mathit{Strict}_{\mathcal{T}}$ be two sets of typicality inclusions. $S' \prec S$ (S'* is preferred to *S) if and only if there is a rank h ($0 \leq h \leq n$) such that, $S_h \subset S'_h$ and, for all ranks $j > h$, $S'_j = S_j$.*

The meaning of $S' \prec S$ is that, considering the highest rank h in which S and S' do not contain the same defeasible inclusions, S' contains more defeasible inclusions in D_h than S.

The preference relation $\prec$ introduced above differs from the one used in the lexicographic closure as the lexicographical order in [51, 23] considers the size of the sets of defaults for each rank. Here, the comparison of the sets of defeasible inclusions with the same rank is based on subset inclusion ($S_h \subset S'_h$) and on equality among sets ($S'_j = S_j$) rather than on the comparison of the size of the sets ($|S_h| < |S'_h|$) and on their equality in size ($|S'_j| = |S_j|$), as in the lexicographic closure. For this reason, the partial order relation $\prec$ introduced above is not necessarily modular, which fits with the fact that in BP-interpretations, the partial order relation $<$ is not required to be modular.

Definition 5.2 ([34]) *Let $\mathcal{T}$ be a TBox, B a concept such that $\mathit{rank}(B) = k$ and $S \subseteq \mathcal{T} \backslash \mathit{Strict}_{\mathcal{T}}$. S is a* maximal set of defeasible inclusions compatible with B in $\mathcal{T}$

[5] As before we can ignore the defeasible inclusions with infinite rank when we consider a set of defaults maximally compatible with a concept B (with rank k) and with E_K, as all defeasible inclusions with infinite rank already belong to E_k.

if:
• $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq \neg B$ [6] *and*
• *there is no* $S' \subseteq \mathcal{T} \backslash Strict_{\mathcal{T}}$ *such that* $S' \prec S$ *and* $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S}' \sqsubseteq \neg B$.

Remember that $\tilde{S}$ is the materialization of S, i.e., $\tilde{S} = \sqcap\{(\neg C \sqcup D) \mid \mathbf{T}(C) \sqsubseteq D \in S\}$. Informally, S is a maximal set of defeasible inclusions compatible with B in $\mathcal{T}$ if there is no set of typicality inclusions S' which is consistent with B and E_k and is preferred to S since it contains more specific defeasible inclusions. To check if a subsumption $\mathbf{T}(B) \sqsubseteq D$ is in the MP-closure of TBox we consider all the maximal sets of defeasible inclusions S that are compatible with B.

Definition 5.3 ([34]) *A query* $\mathbf{T}(B) \sqsubseteq D$ is in the MP-closure of $\mathcal{T}$ *if either the rank of concept* B *in the rational closure of* $\mathcal{T}$ *is infinite or* $rank(B) = k$ *is finite and for all the maximal sets of defeasible inclusions* S *that are compatible with* B *in* $\mathcal{T}$*, we have:* $E_k \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq (\neg B \sqcup D)$.

The construction is similar to the lexicographic closure [51, 23] although, as said above, the lexicographic order $\prec$ used here is different. In particular, for propositional logic, we have proved in [35] that the MP-closure is weaker than Lehmann's lexicographic closure, and stronger than the Relevant Closure [17]. This result, however, cannot be lifted from the propositional case to the description logic $\mathcal{ALC}$, as the lexicographic closure in [23] and the Relevant Closure in [17] are based on a slightly different RC constructions for $\mathcal{ALC}$, compared with the MP-closure defined here, and exploit $\mathcal{ALC}$ entailment over a materialization of the knowledge base rather than $\mathcal{ALC} + \mathbf{T}_R$ entailment (as Definition 5.3 above). While we refer to [40] for a discussion of different rational closure constructions for $\mathcal{ALC}$, Example 5.9 below shows that the lexicographic closure is not weaker than the MP-closure.

Verifying whether a query $\mathbf{T}(B) \sqsubseteq D$ is in the MP-closure of a TBox $\mathcal{T}$, in the worst case, requires to consider a (possibly) exponential number of maximal subsets S of defeasible inclusions compatible with B in $\mathcal{T}$ (exponential in the number of typicality inclusions in $\mathcal{T}$). Instead, computing subsumption in the skeptical closure of $\mathcal{T}$, only requires (for each concept B) a polynomial number (in the size of $\mathcal{T}$) of calls to entailment checks in $\mathcal{ALC} + \mathbf{T}_R$, which can be computed by a linear encoding into $\mathcal{ALC}$ [33].

To reconcile the definition of the MP-closure with the definition of the skeptical closure in Section 3, we prove the following lemma.

Lemma 5.4 *Given a TBox* $\mathcal{T}$*, for any rank* k *in the rational closure of* $\mathcal{T}$*, and for any set* H *of defeasible inclusions in* $\mathcal{T}$*:*

$$E_k \cup H \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg B \iff E_k \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{H} \sqsubseteq \neg B.$$

[6] We keep the formulation used in [34], as it is better suited for proving the correspondence with the BP-semantics. Although $\mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq \neg B$ is not in the language of inclusions in the knowledge base, it is equivalent to $\mathbf{T}(\top) \sqsubseteq \neg\tilde{S} \sqcup \neg B$. We will see below, in Lemma 5.4, that $E_k \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq \neg B$ is also equivalent to $E_k \cup S \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg B$.

Proof 5.5 *($\Leftarrow$) By contraposition, suppose $E_k \cup H \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg B$. Then, there is an $\mathcal{ALC} + \mathbf{T}_R$ model $\mathcal{M} = \langle \Delta, <, I \rangle$ of $E_k \cup H$, and a domain element $x \in \Delta$ such that $k_{\mathcal{M}}(x) = 0$ and $x \in B^I$.*

We show that $x \in \tilde{H}^I$. Let us consider any typicality inclusion $\mathbf{T}(C) \sqsubseteq D$ in H. We show that x is an instance of its materialization $\neg C \sqcup D$, i.e., $x \in (\neg C \sqcup D)^I$. If $x \notin C^I$, the conclusion follows trivially. If $x \in C^I$, considering that x has rank 0 in $\mathcal{M}$ and that $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$, x is a typical C element and hence it must be $x \in D^I$. Therefore, $x \in (\neg C \sqcup D)^I$. As this holds for all the typicality inclusion in H, $x \in \tilde{H}^I$ and, hence, $x \in (\mathbf{T}(\top) \sqcap \tilde{H} \sqcap B)^I$, which proves the thesis.

($\Rightarrow$) By contraposition, let $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{H} \sqsubseteq \neg B$. Then, there is a model $\mathcal{M}_1 = \langle \Delta_1, <_1, I_1 \rangle$ of E_k, and a domain element $x \in \Delta_1$ such that $x \in (\mathbf{T}(\top) \sqcap \tilde{H} \sqcap B)^{I_1}$, i.e., $k_{\mathcal{M}_1}(x) = 0$, $x \in \tilde{H}^{I_1}$ and $x \in B^{I_1}$.

The model $\mathcal{M}_1$ might not satisfy all the typicality inclusions $\mathbf{T}(C) \sqsubseteq D$ in H. Let us consider a model $\mathcal{M}$ of $E_k \cup H$. Such a model must exist, otherwise, the TBox $\mathcal{T}$ would be unsatisfiable and any concept would have an infinite rank in the rational closure of $\mathcal{T}$. Conversely, we know that B has a finite rank k. Hence, let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a finite minimal canonical model of $E_k \cup H$. Existence of a finite, minimal, canonical models of a consistent TBox in $\mathcal{ALC} + \mathbf{T}_R$ is guaranteed by Theorem 7 in [40]). Suppose that Δ and Δ_1 are disjoint. We build from $\mathcal{M}$ and $\mathcal{M}_1$ a new model $\mathcal{M}'$ of $E_k \cup H$ in which the concept $\mathbf{T}(\top) \sqcap B$ is satisfiable.

Let us define $\mathcal{M}' = \langle \Delta', <', I' \rangle$ as follows: $\Delta' = \Delta \cup \Delta_1$; I' is defined on the elements of Δ as I in $\mathcal{M}$, and on the elements of Δ_1 as I_1 in $\mathcal{M}_1$. For the interpretation of concepts: for $x \in \Delta$, $x \in C^{I'}$ if and only if $x \in C^I$; for $x \in \Delta_1$, $x \in C^{I'}$ if and only if $x \in C^{I_1}$. For the interpretation of roles $R \in N_R$: for $x, y \in \Delta$, $(x, y) \in R^{I'}$ if and only if $(x, y) \in R^I$; for $x, y \in \Delta_1$, $(x, y) \in R^{I'}$ if and only if $(x, y) \in R^{I_1}$; and, for any two elements $x \in \Delta$ and $y \in \Delta_1$, $(x, y) \notin R^{I'}$ and $(y, x) \notin R^{I'}$. For all individual constants $a \in N_I$, we let $a^{I'} = a^I$. Finally, for all $w \in \Delta$, we let $k_{\mathcal{M}'}(w) = k_{\mathcal{M}}(w)$, for the element $x \in \Delta_1$ (which is an instance of $\mathbf{T}(\top) \sqcap \tilde{H} \sqcap B$), we let $k_{\mathcal{M}'}(x) = 0$; finally, for all $y \in \Delta_1$ ($y \neq x$), we let $k_{\mathcal{M}'}(y) = n + 1 + k_{\mathcal{M}_1}(y)$, where n is the highest rank value of $k_{\mathcal{M}}$ in $\mathcal{M}$ (n is finite as each element in $\mathcal{M}$ has a finite rank).

It is easy to show that by construction the resulting model $\mathcal{M}'$ satisfies $E_k \cup H$. Let $C \sqsubseteq D$ be strict inclusion in $E_k \cup H$. Let $y \in C^{I'}$. There are two cases: either $y \in \Delta$ or $y \in \Delta_1$. In the first case, $y \in C^I$ in $\mathcal{M}$. As $\mathcal{M}$ satisfies K, $y \in D^I$ and, by definition of $\mathcal{M}'$, $y \in D^{I'}$. In the second case, $y \in C^{I_1}$. As $\mathcal{M}_1$ satisfies all the strict inclusions in $\mathcal{T}$ (which belong to E_k), $y \in D^{I_1}$ and, by definition of $\mathcal{M}'$, $y \in D^{I'}$.

Let $\mathbf{T}(C) \sqsubseteq D$ be a defeasible inclusion in $E_k \cup H$. If $rank(C) \geq k$, then by the construction of the rational closure $\mathbf{T}(C) \sqsubseteq D$ is in E_k and hence is satisfied both in $\mathcal{M}$ and in $\mathcal{M}_1$. Let $z \in (\mathbf{T}(C))^{I'}$, then either $z \in \Delta$ or $z \in \Delta_1$. In the first case, z is C-minimal in $\mathcal{M}$ and $z \in D^I$. Hence, by definition of $\mathcal{M}'$, $z \in D^{I'}$. In the second case, z is C-minimal in $\mathcal{M}_1$ and $z \in D^{I_1}$. Hence, by definition of $\mathcal{M}'$, $z \in D^{I'}$.

If $rank(C) = j < k$, then $\mathbf{T}(C) \sqsubseteq D$ is in H but not in E_k. As the rank of C in the rational closure is finite, by Proposition 13 in [40] (recalled in Section 2), C has a finite rank j in any minimal canonical model of the TBox $\mathcal{T}$. Hence, C is consistent with the TBox $\mathcal{T}$, as well as with its subset $E_k \cup H \subseteq \mathcal{T}$. As $\mathcal{M}$ is a canonical model of $E_k \cup H \subseteq \mathcal{T}$, there must be an element in $w \in \Delta$ such that $w \in C^I$. Therefore, *each minimal C element in $\mathcal{M}$ either is x (and, in this case, x is in $(\neg C \sqcap D)^{I'}$ and hence in $D^{I'}$), or it is an element $z \in \Delta$. As $\mathcal{M}$ satisfies H, it satisfies $\mathbf{T}(C) \sqsubseteq D$ and, hence, $z \in D$.*

From this, we can conclude that $\mathcal{M}'$ is a model satisfying $E_k \cup H$, which contains an element x with rank $k_{\mathcal{M}'}(x) = 0$ such that $x \in B$. Therefore, $E_k \cup H \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg B$, which concludes the proof.

Example 5.6 *Let us consider again the knowledge base $K = (\mathcal{T}, \mathcal{A})$ of Example 4.11, with $\mathcal{A} = \emptyset$ and the following TBox $\mathcal{T}$:*
$\mathbf{T}(\mathit{Student}) \sqsubseteq \mathit{Young}$
$\mathbf{T}(\mathit{Student}) \sqsubseteq \neg \mathit{PayTaxes} \sqcap \exists \mathit{hasSSN}.\top$
$\mathbf{T}(\mathit{Employee}) \sqsubseteq \mathit{PayTaxes} \sqcap \exists \mathit{hasSSN}.\top$
$\mathbf{T}(\mathit{Student} \sqcap \mathit{Employee}) \sqsubseteq \neg\, \mathit{Young}$
We have seen that the typicality inclusion $\mathbf{T}(\mathit{Student} \sqcap \mathit{Employee}) \sqsubseteq \exists \mathit{hasSSN}.\top$ is not in the skeptical closure of $\mathcal{T}$, but it holds in all the minimal canonical BP-models of K. We can see that $\mathbf{T}(\mathit{Student} \sqcap \mathit{Employee}) \sqsubseteq \exists \mathit{hasSSN}.\top$ follows from the MP-closure of TBox $\mathcal{T}$. In fact, in this example there are two maximal sets of defeasible inclusions compatible with $B = \mathit{Student} \sqcap \mathit{Employee}$ (where $\mathit{rank}(B) = 1$):
$S = \{\mathbf{T}(\mathit{Student}) \sqsubseteq \neg \mathit{PayTaxes} \sqcap \exists \mathit{hasSSN}.\top,\ \mathbf{T}(\mathit{Student} \sqcap \mathit{Employee}) \sqsubseteq \neg\, \mathit{Young}\}$
$S' = \{\mathbf{T}(\mathit{Employee}) \sqsubseteq \mathit{PayTaxes} \sqcap \exists \mathit{hasSSN}.\top,\ \mathbf{T}(\mathit{Student} \sqcap \mathit{Employee}) \sqsubseteq \neg\, \mathit{Young}\}$
S is partitioned, according to the ranks of defaults, as follows:
$S_0 = \{\mathbf{T}(\mathit{Student}) \sqsubseteq \neg \mathit{PayTaxes} \sqcap \exists \mathit{hasSSN}.\top\}$;
$S_1 = \{\mathbf{T}(\mathit{Student} \sqcap \mathit{Employee}) \sqsubseteq \neg\, \mathit{Young}\}$; $S_2 = \emptyset$
and S' is partitioned as follows:
$S'_0 = \{\mathbf{T}(\mathit{Employee}) \sqsubseteq \mathit{PayTaxes} \sqcap \exists \mathit{hasSSN}.\top\}$;
$S'_1 = \{\mathbf{T}(\mathit{Student} \sqcap \mathit{Employee}) \sqsubseteq \neg\, \mathit{Young}\}$; $S'_2 = \emptyset$.
Observe that neither $S \prec S'$ nor $S' \prec S$ and hence both S and S' are maximal sets of defeasible inclusions compatible with B. As $E_k \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq (\neg(\mathit{Student} \sqcap \mathit{Employee}) \sqcup \exists \mathit{hasSSN}.\top)$, and the same holds for for S', it follows that $\mathbf{T}(\mathit{Student} \sqcap \mathit{Employee}) \sqsubseteq \exists \mathit{hasSSN}.\top$ is in the MP-closure of $\mathcal{T}$. In this case, S and S' correspond to the bases of the lexicographic closure of $\mathcal{T}$.

It is easy to see that any defeasible inclusion in the skeptical closure of $\mathcal{T}$ is as well in its MP-closure.

Proposition 5.7 *$\mathbf{T}(B) \sqsubseteq D$ is in the skeptical closure of a TBox $\mathcal{T}$, then $\mathbf{T}(B) \sqsubseteq D$ is in the MP-closure of $\mathcal{T}$.*

Proof 5.8 *(Sketch) Suppose $\mathbf{T}(B) \sqsubseteq D$ is in the skeptical closure of $\mathcal{T}$. Then, either $\mathit{rank}(B) = \infty$ in the rational closure construction of $\mathcal{T}$ and, hence, $\mathbf{T}(B) \sqsubseteq D$ is in the MP-closure, or $\mathit{rank}(B)$ is finite and $S^{sk,B} \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg B \sqcup D)$, where $S^{sk,B} = E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S^B_h$. We show that $S^{sk,B} \subseteq S$ for each maximal set S of defeasible inclusions compatible with B in $\mathcal{T}$.*

We prove that $S^B_{k-j} \subseteq S$ for all $j = 1, \ldots, k-h$, by induction on j. For $j = 1$, we show that $S^B_{k-1} \subseteq S$. By the global compatibility condition, $E_k \cup S^B_{k-1} \not\models_{\mathcal{ALC}+\mathbf{T}_R}$ $\mathbf{T}(\top) \sqsubseteq \neg B$ and all defeasible inclusions with rank k not in S^B_{k-1} are individually incompatible with B wrt. E_k (and none of them can belong to S). By Lemma 5.4, $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S}^B_{k-1} \sqsubseteq \neg B$. Also, by maximality of S, it is easy to see that it cannot be the case that some defeasible inclusion in S^B_{k-1} does not belong to S. The inductive step is similar. As all S^B_{k-j} are included in S, $E_k \cup S \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq$ $(\neg B \sqcup D)$ and, by Lemma 5.4, $E_k \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq (\neg B \sqcup D)$. As this is true for each maximal set S of defeasible inclusions compatible with B in $\mathcal{T}$, $\mathbf{T}(B) \sqsubseteq D$ is in the MP-closure of a TBox $\mathcal{T}$.

From Proposition 5.7 and Example 5.6 it follows that the skeptical closure is strictly weaker than the MP-closure. Before establishing a correspondence between the MP-closure and BP-semantics, let us show that the lexicographic closure allows conclusions which are not in the MP-closure.

Example 5.9 *If we modify the knowledge base in Example 5.6 above, by adding to the TBox the typicality inclusion $\mathbf{T}(Student) \sqsubseteq \neg PayTaxes \sqcap Smart$ we would get again two maximal sets of defeasible inclusions compatible with $B = Student \sqcap Employee$ in the MP-closure construction:*

$S = \{\mathbf{T}(Student) \sqsubseteq \neg PayTaxes \sqcap \exists hasSSN.\top,$
$\quad \mathbf{T}(Student) \sqsubseteq \neg PayTaxes \sqcap Smart, \mathbf{T}(Student \sqcap Employee) \sqsubseteq \neg Young\}$

$S' = \{\mathbf{T}(Employee) \sqsubseteq PayTaxes \sqcap \exists hasSSN.\top,$
$\quad \mathbf{T}(Student \sqcap Employee) \sqsubseteq \neg Young\}$

As $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S'} \sqsubseteq (\neg(Student \sqcap Employee) \sqcup \neg PayTaxes \sqcap Smart)$, then $\mathbf{T}(Student \sqcap Employee) \sqsubseteq \neg PayTaxes \sqcap Smart$ is not in the MP-closure of $\mathcal{T}$. As a difference, only S corresponds to a base in the lexicographic closure, as S contains two defaults with rank 0 and one with rank 1, while S' contains just one default with rank 0 and one with rank 1. The lexicographic closure can then conclude that employed students do not pay taxes and are smart.

To show that the typicality inclusions derivable form the MP-closure of a TBox $\mathcal{T}$ are exactly those that hold in all its minimal canonical BP-models, we prove the next two propositions. The first one shows that the MP-closure is sound with respect to the minimal canonical BP-semantics: If $\mathbf{T}(B) \sqsubseteq D$ is in the MP-closure of $\mathcal{T}$, then $\mathcal{T} \models^{min}_{BP} \mathbf{T}(B) \sqsubseteq D$. Let us prove the contrapositive.

Proposition 5.10 *Let $K = (\mathcal{T}, \mathcal{A})$ be a knowledge base and B a concept with a finite rank $rank(B) = k$ in the rational closure of $\mathcal{T}$. If there is a minimal canonical BP-model $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ of K and an element $x \in \Delta$ such that $x \in min_<(B^I)$ and $x \notin D^I$, then there is a maximal set of defeasible inclusions S compatible with B in $\mathcal{T}$, such that $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq (\neg B \sqcup D)$.*

Proof 5.11 *Assume that for some minimal canonical BP-model $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ of K there is an element $x \in \Delta$ such that $x \in min_<(B^I)$ and $x \notin D^I$. We construct S as the set of all the defeasible inclusions in TBox which are not violated in x, i.e.*

$$S = \{\mathbf{T}(C) \sqsubseteq E \in \mathit{TBox} \mid \ x \in (\neg C \sqcup E)\},$$

and we show that, S is an MP-basis for B and $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq (\neg B \sqcup D)$.

Let $\mathcal{M}^{RC} = \langle \Delta, <_{rc}, I \rangle$. By construction, $\mathcal{M}^{RC}$ is a minimal canonical model of the rational closure of K. By Proposition 12 in [40], $\mathcal{M}_k^{RC}$ (i.e. the model obtained by $\mathcal{M}^{RC}$ by collapsing all the elements with rank $\leq k$ to rank 0, and updating each other rank $k+i$ to i) satisfies E_k, that is, $\mathcal{M}_k^{RC} \models_{\mathcal{ALC}+\mathbf{T}_R} E_k$. Also, as $rank(B) = k$ and $x \in \mathbf{T}(B)^I$, x must have rank k in $\mathcal{M}^{RC}$, and hence rank 0 in $\mathcal{M}_k^{RC}$ (and, clearly, $k_{\mathcal{M},rc}(x) = k$ in $\mathcal{M}$). Thus, $x \in \mathbf{T}(\top)^I$ holds in $\mathcal{M}_k^{RC}$, but also $x \in B^I$ and $x \in \tilde{S}^I$ (by definition of S). Therefore $\mathcal{M}_k^{RC} \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq \neg B$. Hence, $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq \neg B$, i.e. S is a set of defeasible inclusions compatible with B.

Observe that, as $x \notin D^I$, $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqcap \neg D \sqsubseteq \neg B$, then $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq (\neg B \sqcup D)$, i.e., if S is an MP-basis for B, $\mathbf{T}(B) \sqsubseteq D$ does not follow from the MP-closure of TBox. To see that S is an MP-basis for B, we have to show that S is a maximal set of defeasible inclusions compatible with B in $\mathcal{T}$. Suppose, for a contradiction, that there is a set S' such that $S' \prec S$ and $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S}' \sqsubseteq \neg B$. Then, there must be a $\mathcal{ALC} + \mathbf{T}_R$ model $\mathcal{N} = \langle \Delta', <'_{rc} \ I' \rangle$ of E_k and an element $y \in \Delta'$, having rank 0 in $\mathcal{N}$ such that: $y \in (\tilde{S}' \sqcap B)^{I'}$.

As $\mathcal{M}$ is canonical, then $\mathcal{M}^{RC}$ is canonical as well. Hence, there must be an element $z \in \Delta$ such that $z \in (\tilde{S}' \sqcap B)^I$ (i.e., the interpretation of all non-extended concepts in z is the same as in y in $\mathcal{N}$). As y has rank 0 in $\mathcal{N}$, y satisfies all the defeasible inclusions in E_k. Hence, the concept $\tilde{S}' \sqcap B$ must have rank k in the rational closure and, therefore, z must have rank k in $\mathcal{M}^{RC}$. Thus, $z \in (\mathbf{T}(\top) \sqcap \tilde{S}' \sqcap B)^I$ in $\mathcal{M}_k^{RC}$, and, clearly, $k_{\mathcal{M},rc}(z) = k$ in $\mathcal{M}$.

Since $S' \prec S$ there must be some h such that, $S_h \subset S'_h$ and, for all $j > h$, $S'_j = S_j$. Thus, there is some defeasible inclusion $\mathbf{T}(C') \sqsubseteq E' \in S'$ with rank h in RC, such that $\mathbf{T}(C') \sqsubseteq E' \notin S$. Therefore, z does not violate $\mathbf{T}(C') \sqsubseteq E'$ (i.e., $z \in (\neg C' \sqcup E')^I$), while x violates it (i.e., $x \in (C' \sqcap \neg E')^I$). On the other hand, all the defeasible inclusions violated by z and not by x cannot have a rank $\geq h$ in RC, as x satisfies only the inclusions S (by construction of S) and, for all $j \geq h$, $S'_j = S_j$ (the typicality inclusions with infinite rank are trivially satisfied both in x and in z). Therefore, $z < x$ holds in $\mathcal{M}$ by condition (3), and x cannot be a typical B element, thus contradicting the hypothesis.

The next proposition shows that the MP-closure is complete with respect to the minimal canonical BP-semantics: If $\mathcal{T} \models_{BP}^{min} \mathbf{T}(B) \sqsubseteq D$, then $\mathbf{T}(B) \sqsubseteq D$ is in the MP-closure of the TBox $\mathcal{T}$. Let us prove the contrapositive.

Proposition 5.12 *Let $K = (\mathcal{T}, \mathcal{A})$ be a knowledge base and $\mathbf{T}(B) \sqsubseteq D$ a query such that B has a finite rank in RC. If $\mathbf{T}(B) \sqsubseteq D$ is not in the MP-closure of $\mathcal{T}$, then there is a minimal canonical MP-model $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ of K and an element $x \in \Delta$ such that $x \in min_<(B^I)$ and $x \notin D^I$.*

Proof 5.13 *Let $rank(B) = k$ in RC. If $\mathbf{T}(B) \sqsubseteq D$ is not in the MP-closure of $\mathcal{T}$, then there is a maximal set of defeasible inclusions S compatible with B in $\mathcal{T}$, such that $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq (\neg B \sqcup D)$. Then*

$$E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg(\tilde{S} \sqcap B \sqcap \neg D)$$

and concept $\tilde{S} \sqcap B \sqcap \neg D$ is not exceptional with respect to E_k and, in the rational closure of $\mathcal{T}$, it must have rank less or equal to k. As $rank(B) = k$, it must be $rank(\tilde{S} \sqcap B \sqcap \neg D) = k$.

Let us consider any minimal canonical $\mathcal{ALC} + \mathbf{T}_R$ model $\mathcal{N} = \langle \Delta', <_{RC}, I' \rangle$ of K. As $rank(\tilde{S} \sqcap B \sqcap \neg D) = k$, by Proposition 13 in [40] (recalled in Section 2), the concept $\tilde{S} \sqcap B \sqcap \neg D$ must have rank k in any minimal canonical model of K. Therefore, $k_{\mathcal{N}}(\tilde{S} \sqcap B \sqcap \neg D) = k$, and there is an element $y \in \Delta$ such that $y \in (\tilde{S} \sqcap B \sqcap \neg D)^{I'}$ and $k_{\mathcal{N}}(y) = k$.

From $\mathcal{N}$ we build a minimal canonical MP-model $\mathcal{M} = \langle \Delta, <_{rc}, <, I \rangle$ of K falsifying $\mathbf{T}(B) \sqsubseteq D$ as follows. We let $\Delta = \Delta'$, $I = I'$, $<_{rc} = <_{RC}$ and we define $<$ as follows:

$x < y$ **iff** (i) *y violates some defeasible inclusion $\mathbf{T}(C) \sqsubseteq D \in \mathcal{T}$ satisfied by x and*
(ii) *for all $\mathbf{T}(C_j) \sqsubseteq D_j \in \mathcal{T}$, which is violated by x and not by y,*
there is a $\mathbf{T}(C_k) \sqsubseteq D_k \in \mathcal{T}$, which is violated by y and not by x,
such that $k_{\mathcal{M},rc}(C_j) < k_{\mathcal{M},rc}(C_k)$.

Observe that, for all concepts C, $k_{\mathcal{M},rc}(C) = k_{RC}(C) = rank(C)$, the rank of C in the rational closure. We have to show that $\mathcal{M}$ is a minimal canonical MP-model of $\mathcal{T}$ and that $y \in (\mathbf{T}(B) \sqcap \neg D)^I$.

We first show that $\mathcal{M}$ is an MP-model of K, that it is canonical and that it is minimal among the canonical MP-models of K. To show that $\mathcal{M}$ is an MP-model of K, we observe that, by definition of $<$, condition (3) in Definition 4.2 holds for $\mathcal{M}$ by construction.

It can be easily seen that $<$ is an irreflexive, asymmetric and transitive relation. Also, $\mathcal{M}$ satisfies all the assertions in $\mathcal{A}$ and all the strict inclusions $E \sqsubseteq F$ in $\mathcal{T}$, since $\mathcal{N}$ does, $\Delta = \Delta'$ and $I = I'$. To see that $\mathcal{M}$ is an MP-model of $\mathcal{T}$, we have also to show that for all $\mathbf{T}(E) \sqsubseteq F$ in $\mathcal{T}$, $min_{<_{rc}}(E^I) \subseteq F^I$ holds. This follows from the fact that $min_{<_{RC}}(E^{I'}) \subseteq F^{I'}$ holds in $\mathcal{N}$ and that, by definition of $\mathcal{M}$, $<_{rc} = <_{RC}$ and $I = I'$.

We show that $\mathcal{M}$ is a canonical BP-model of K: If not, there are $C_1, C_2, \ldots, C_n$ such that $\mathcal{T} \not\models_{BP} C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n \sqsubseteq \bot$, but there is no $x \in \Delta$ such that $x \in (C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n)^I$. As, by Theorem 4.7, $\mathcal{T} \not\models_{\mathcal{ALC}+\mathbf{T}_R} C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n \sqsubseteq \bot$, this would contradict the hypothesis that $\mathcal{N}$ is an $\mathcal{ALC} + \mathbf{T}_R$ canonical model of $\mathcal{T}$.

We prove the minimality of $\mathcal{M}$. If, by absurdum, $\mathcal{M}$ were not a minimal canonical BP model, then there would be a BP model $\mathcal{M}'' = \langle \Delta'', <''_{rc}, <'', I'' \rangle$ in $Min_{RC}(K)$, such that $\Delta'' = \Delta$, $I'' = I$, and $\mathcal{M}'' \prec_{BP} \mathcal{M}$. Observe that the ranking function in $\mathcal{M}''$ must be the same as in $\mathcal{M}$ (i.e., $k_{\mathcal{M}'',rc}(y) = k_{\mathcal{M},rc}(y)$ for all $y \in \Delta$) as it is determined by a minimal canonical $\mathcal{ALC} + \mathbf{T}_R$ model of K (and hence by the rational closure of TBox $\mathcal{T}$).

Concerning $<''$, as $\mathcal{M}''$ is an BP interpretation, $<''$ must satisfy condition (3) in Definition 4.2. If $x < y$ in $\mathcal{M}$ then (i) and (ii) hold by construction of $\mathcal{M}$ and, then, $x <'' y$ by (3), since $k_{\mathcal{M}'',rc}(C) = k_{\mathcal{M},rc}(C)$ for all concepts C. Hence, $< \subseteq <''$, which contradicts the hypothesis that that $\mathcal{M}'' \prec_{BP} \mathcal{M}$.

To conclude the proof, we want to show that $y \in (\mathbf{T}(B) \sqcap \neg D)^I$. We have seen that in $\mathcal{N}$ there is an element $y \in \Delta$ such that $y \in (\tilde{S} \sqcap B \sqcap \neg D)^{I'}$ and $k_{\mathcal{N}}(y) = k$.

By construction of $\mathcal{M}$, $I = I'$ and then $y \in (B \sqcap \neg D)^I$. Furthermore, $<_{rc}=<_{RC}$ and, hence, $k_{\mathcal{M},rc}(y) = k_{\mathcal{N}}(y) = k$ and, also, $k_{\mathcal{M},rc}(B) = k_{\mathcal{N}}(B) = rank(B) = k$.

To see that $y \in min_<(B)$, it sufficies to show that there is no $z \in \Delta$ such that $z \in B^I$ and $z < y$. Suppose for a contradiction that there is such a z. As z is a B-element, it cannot have rank less than k in the rational closure (it cannot be $k_{\mathcal{M},rc}(z) = j < k = rank(B)$). Hence, $k_{\mathcal{M},rc}(z) \geq k$.

If $k_{\mathcal{M},rc}(z) > k$, then $y <_{rc} z$ and, by Proposition 4.3, $y < z$, a contradiction with the hypothesis that $z < y$. Then $k_{\mathcal{M},rc}(z) = k$. Let S' be the set of defeasible inclusions not violated by z, i.e., $S' = \{\mathbf{T}(C) \sqsubseteq E \in TBox \mid \ z \in (\neg C \sqcup E)\}$. Then $z \in (\tilde{S'} \sqcap B)^I$. Let $\mathcal{M}^{RC} = \langle \Delta, <_{rc}, I \rangle$ be the $\mathcal{ALC} + \mathbf{T}_R$ model obtained from $\mathcal{M}$, ignoring the preference relation $<$. As $\mathcal{M}^{RC}$ is a minimal canonical $\mathcal{ALC} + \mathbf{T}_R$ model of K, by Proposition 12 in [40], $\mathcal{M}^{RC}_k \models_{\mathcal{ALC}+\mathbf{T}_R} E_k$ and, as $k_{\mathcal{M},rc}(z) = k$, z must have rank 0 in $\mathcal{M}^{RC}_k$. Therefore, $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S'} \sqsubseteq \neg B$.

As $z < y$, for all defeasible inclusions $\mathbf{T}(C_j) \sqsubseteq A_j \in \mathcal{T}$ violated by z and not by y, there is a more specific defeasible inclusion $\mathbf{T}(C_k) \sqsubseteq A_k \in \mathcal{T}$ violated by y and not by z (that is $k_{\mathcal{M},rc}(C_j) < k_{\mathcal{M},rc}(C_k)$). Suppose that j is the rank of the defeasible inclusion with highest rank violated by z and that h is the rank of the defeasible inclusion with highest rank violated by y. It must be $j < h$. Therefore, $S_h \subset S'_h$ (as z satisfies all the defeasible inclusions with rank $h > j$). Therefore, S' is preferred to S, $S' \prec S$. However, this contradicts the hypothesis that S is a maximal set of defeasible inclusions compatible with B in $\mathcal{T}$. Therefore, a z with $z < y$ cannot exist and $y \in \mathbf{T}(B)^I$, so that $y \in (\mathbf{T}(B) \sqcap \neg D)^I$.

We can now establish a correspondence between the minimal canonical BP semantics and the MP-closure.

Theorem 5.14 *Given a knowledge base $K = (\mathcal{T}, \mathcal{A})$ and a query $\mathbf{T}(B) \sqsubseteq D$, $\mathcal{T} \models^{min}_{BP} \mathbf{T}(B) \sqsubseteq D$ if and only if $\mathbf{T}(B) \sqsubseteq D$ follows from the MP-closure of the TBox $\mathcal{T}$.*

Proof 5.15 *The proof of this result can be done by contraposition and is an easy consequence of Proposition 5.10 and Proposition 5.12. Just observe that, for the "If" part, when $\mathcal{T} \not\models^{min}_{BP} \mathbf{T}(B) \sqsubseteq D$, concept B must have a finite rank, otherwise $\mathbf{T}(B) \sqsubseteq D$ would be a logical consequence of $\mathcal{T}$, for any concept D. For the "Only if" part, when $\mathbf{T}(B) \sqsubseteq D$ does not follow from the MP-closure of the TBox $\mathcal{T}$, the rank of B in the rational closure must be finite.*

In the propositional case, as a consequence of the property proved by Lehmann and Magidor [50] that any preferential model defines a preferential consequence relation, it has been shown that the MP-closure defines a preferential consequence relation [35]. The KLM properties of preferential consequence have been lifted to $\mathcal{ALC}$ knowledge bases by Britz et al. in [16], where formulation of the semantic properties in terms of defeasible inclusions is provided, while formulations for typicality inclusions can be found in [38, 31]. Based on the BP-semantics, which is a preferential semantics, it is immediate to see that the MP-closure for $\mathcal{ALC}$ defines a preferential consequence relation. Instead, the MP-closure violates the property of Rational Monotonicity already in the propositional case (we refer to Example 5 in [35], which is a reformulation of Lehmann's musician example [51]). In the next section, we will prove that the skeptical closure satisfies the KLM properties of a preferential consequence relation as well.

6 A semantic characterization for the skeptical closure

We have already seen that the skeptical closure is weaker than the MP-closure (Proposition 5.7) and that, for a B with rank k, any maximal set S of defeasible inclusions compatible with B in $\mathcal{T}$ must contain all the defeasible inclusions in $S^{sk,B}$. Indeed, using Lemma 5.4, one can equivalently reformulate the notion of global compatibility of a set of defeasible inclusions (Definition 3.2) as follows.

Proposition 6.1 *For a TBox $\mathcal{T}$, a concept B with finite $rank(B) = k$ and sets S and S' of defeasible inclusions,* S is globally compatible with B w.r.t. $E_k \cup S'$ *iff* $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqcap \tilde{S}' \sqsubseteq \neg B$.

This reformulation makes the relationship between the notions of skeptical closure and of MP-closure more evident. In particular, when in the MP-closure construction there is a unique maximal set of defeasible inclusions S compatible with B in $\mathcal{T}$, i.e., such that $E_k \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqcap \tilde{S} \sqsubseteq \neg B$, then $E_k \cup S$ corresponds to the skeptical closure $S^{sk,B}$ of $\mathcal{T}$ with respect to B.

When in the MP-closure there are different maximal sets of defeasible inclusions $S^1, \ldots, S^r$ compatible with B in $\mathcal{T}$, the skeptical closure is defined to contain, in addition to E_k, the defeasible inclusions with rank j in $S^1, \ldots, S^r$, for those ranks j from h to $k-1$ on which $S^1, \ldots, S^r$ exactly agree (i.e., $S^1_j = \ldots = S^r_j$), where $h-1$ is the highest rank on which $S^1, \ldots, S^r$ disagree (i.e., $S^l_{h-1} \neq S^m_{h-1}$, for some l and m). If the sets $S^1, \ldots, S^r$ disagree on some defeasible inclusion with rank j, no defeasible inclusion with rank j or lower is included in the skeptical closure.

Based on the reformulation above and on the correspondence between the MP-closure of a knowledge base and its minimal canonical BP-models in Section 5, we are now able to provide a semantic characterization of the skeptical closure.

Given a TBox $\mathcal{T}$, let $DI(B)$ be the set of the defeasible inclusions $\mathbf{T}(C) \sqsubseteq D \in \mathcal{T}$ which are satisfied by all the minimal B elements in any of the minimal canonical BP-models of $\mathcal{T}$:

$DI(B) = \{\mathbf{T}(C) \sqsubseteq D \in K \mid x \in (\neg C \sqcup D)^I$, for any $x \in min_<(B^I)$ in any minimal canonical BP-model $\mathcal{M} = \langle \Delta, <_{rc}, <, \cdot^I \rangle$ of $\mathcal{T}\}$

Let *Confl_DI*(B) be the set of the conflicting defeasible inclusions for B in $\mathcal{T}$, defined as the typicality inclusions which are satisfied in some minimal B element in a minimal canonical BP-model of $\mathcal{T}$, but not in all of them:

$Confl\_DI(B) = \{\mathbf{T}(C) \sqsubseteq D \in K \mid$ for some minimal canonical BP-model $\mathcal{M} = \langle \Delta, <_{rc}, <, \cdot^I \rangle$ of $\mathcal{T}$, there are $x, y \in min_<(B^I)$ such that $x \in (\neg C \sqcup D)^I$ and $y \in (C \sqcap \neg D)^I\}$

The set $Confl\_DI(B)$ refers to the defaults with no agreement among minimal B elements in at least some minimal canonical BP-model of $\mathcal{T}$. We identify the defeasible inclusions with rank j in $DI(B)$ and in *Confl_DI*(B), respectively, as:

$DI_j(B) = DI(B) \cap D_j$ $\qquad$ $Confl\_DI_j(B) = Confl\_DI(B) \cap D_j$

where D_j is the set of all typicality inclusions with rank j in $\mathcal{T}$ (as introduced in Section 5). We can now define the set of defeasible inclusions $DI\_Sk(B)$, which are included in the skeptical closure of B, $S^{sk,B}$, as follows:

$$DI\_Sk(B) = \bigcup_{j=h,k-1} DI_j(B)$$

where h is the lowest integer, form 0 to $k-1$, such that, for all $j \geq h$, $Confl\_DI_j(B) = \emptyset$.

Essentially, $DI\_Sk(B)$ contains the defeasible inclusions on which all the minimal canonical models agree, in the following sense: for each rank j, from h to $k-1$, $DI_j(B)$ is the set of all the defeasible inclusions of rank j which are satisfied by all the minimal B-elements in all the minimal canonical BP-model of $\mathcal{T}$. Also, the minimal B instances of the minimal canonical BP-models of $\mathcal{T}$ must agree on accepting or not all the defeasible inclusions with rank $\geq h$ (as there are no conflicting defeasible inclusions for B with rank $\geq h$). Instead, minimal B instances disagree on accepting or not some defeasible inclusion with rank $h-1$, and no inclusion with rank $h-1$ or lower is included in $DI\_Sk(B)$. Given these considerations, the next result follows from Propositions 5.3 and 6.1.

Proposition 6.2 *Let* $\mathbf{T}(B) \sqsubseteq D$ *be a query and* $\mathcal{T}$ *a TBox. The defeasible inclusion* $\mathbf{T}(B) \sqsubseteq D$ *is in the skeptical closure of* $\mathcal{T}$ iff $Strict_{\mathcal{T}} \cup DI\_Sk(B) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg B \sqcup D)$*, where* $Strict_{\mathcal{T}}$ *is the set of strict inclusions in* $\mathcal{T}$.

Observe that, as each defeasible inclusion $\mathbf{T}(B) \sqsubseteq D$, in $DI\_Sk(B)$ by construction is satisfied in all the minimal B-elements of any minimal canonical BP-model of $\mathcal{T}$, it holds that $\mathcal{T} \models^{min}_{BP} \mathbf{T}(B) \sqsubseteq D$. Then, by Theorem 5.14, $\mathbf{T}(B) \sqsubseteq D$ is in the MP-closure of the TBox $\mathcal{T}$. As we have seen, the skeptical closure is a weaker construction than the MP-closure.

We exploit Property 6.2 to prove that the skeptical closure is a preferential consequence relation.

Proposition 6.3 *The following KLM properties of preferential entailment relation are satisfied by the skeptical closure:*
(LLE) If $A \equiv B$ and $\mathbf{T}(A) \sqsubseteq C$, then $\mathbf{T}(B) \sqsubseteq C$
(RW) If $C \sqsubseteq D$ and $\mathbf{T}(A) \sqsubseteq C$, then $\mathbf{T}(B) \sqsubseteq D$
(REFL) $\mathbf{T}(A) \sqsubseteq A$
(AND) If $\mathbf{T}(A) \sqsubseteq C$ and $\mathbf{T}(A) \sqsubseteq D$, then $\mathbf{T}(A) \sqsubseteq C \sqcap D$
(OR) If $\mathbf{T}(A) \sqsubseteq C$ or $\mathbf{T}(A) \sqsubseteq D$, then $\mathbf{T}(A) \sqsubseteq C \sqcup D$
(CM) If $\mathbf{T}(A) \sqsubseteq D$ and $\mathbf{T}(A) \sqsubseteq C$, then $\mathbf{T}(A \sqcap D) \sqsubseteq C$

Proof 6.4 *We prove that each property is satisfied.*
• (LLE)*: Let* A *and* B *be two equivalent* $\mathcal{ALC} + \mathbf{T}_R$ *concepts. As they are satisfied or violated by the same domain elements,* $DI\_Sk(A) = DI\_Sk(B)$.
Suppose $\mathbf{T}(A) \sqsubseteq C$ *is in the skeptical closure of* $\mathcal{T}$*. By Proposition 6.2,*

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup C)$

Then

$Strict_{\mathcal{T}} \cup DI\_Sk(B) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg B \sqcup C)$

and, hence, $\mathbf{T}(B) \sqsubseteq C$ *is in the skeptical closure of* $\mathcal{T}$.

• (RW): Let C and D be two $\mathcal{ALC} + \mathbf{T}_R$ concepts such that $\models_{\mathcal{ALC}+\mathbf{T}_R} C \sqsubseteq D$. Hence, $\models_{\mathcal{ALC}+\mathbf{T}_R} \top \sqsubseteq \neg C \sqcup D$. Suppose $\mathbf{T}(A) \sqsubseteq C$ is in the skeptical closure of $\mathcal{T}$. By Proposition 6.2,

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup C)$

and then:

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup C) \sqcap (\neg C \sqcup D)$

Thus:

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup D)$.

Hence, $\mathbf{T}(A) \sqsubseteq D$ is in the skeptical closure of $\mathcal{T}$.

• $(REFL)$ $\mathbf{T}(A) \sqsubseteq A$ is in the skeptical closure of $\mathcal{T}$ as, by Proposition 6.2, it is to see that

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup A)$,

which holds for all A, as any propositional tautology is valid in $\mathcal{ALC} + \mathbf{T}_R$.

• (AND): suppose $\mathbf{T}(A) \sqsubseteq C$ and $\mathbf{T}(A) \sqsubseteq D$ are in the skeptical closure of $\mathcal{T}$. By Proposition 6.2,

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup C)$ *and*
$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup D)$

then

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup (C \sqcap D))$.

Hence $\mathbf{T}(A) \sqsubseteq C \sqcap D$ is in the skeptical closure of $\mathcal{T}$.

• (OR) suppose $\mathbf{T}(A) \sqsubseteq C$ is in the skeptical closure of $\mathcal{T}$. By Proposition 6.2,

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup C)$

and then

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup (C \sqcup D))$

and therefore $\mathbf{T}(A) \sqsubseteq C \sqcup D$ is in the skeptical closure of $\mathcal{T}$.

• (CM): suppose $\mathbf{T}(A) \sqsubseteq D$ and $\mathbf{T}(A) \sqsubseteq C$ are in the skeptical closure of $\mathcal{T}$. By Proposition 6.2,

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup D)$ *and*
$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg A \sqcup C)$

As $\neg A \sqcup C \sqsubseteq \neg A \sqcup \neg D \sqcup C$:

$Strict_{\mathcal{T}} \cup DI\_Sk(A) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg(A \sqcap D) \sqcup C)$.

We can prove that $min_<(A^I) = min_<((A \sqcap D)^I)$ in all minimal canonical BM-models of $\mathcal{T}$.

As $\mathbf{T}(A) \sqsubseteq D$ is in the skeptical closure of $\mathcal{T}$, $\mathbf{T}(A) \sqsubseteq D$ is in the MP-closure of $\mathcal{T}$. Then, in all minimal canonical BM-models of $\mathcal{T}$, $min_<(A^I) \subseteq D^I$. Then $min_<(A^I) \subseteq min_<((A \sqcap D)^I)$.

The converse also holds. Assume that $x \in min_<(A \sqcap D)^I$. Suppose for a contradiction that $x \notin min_<(A^I)$. Then, there would be an element z such that $z < x$ and $z \in min_<(A^I)$ (by well-foundedness). But, as $z \in min_<(A^I)$ and $\mathbf{T}(A) \sqsubseteq D$ holds in all the minimal canonical BP-models of $\mathcal{T}$, $z \in D^I$. Therefore, $z \in (A \sqcap D)^I$, which (given $z < x$) contradicts the hypothesis that $x \in min_<((A \sqcap D)^I$. Hence, it cannot be the case that $x \notin min_<(A^I)$, and $min_<((A \sqcap D)^I) \subseteq min_<(A^I)$.

As $min_<(A^I) = min_<((A \sqcap D)^I)$, by construction, it follows that $DI\_Sk(A) = DI\_Sk(A \sqcap D)$, and hence: $Strict_{\mathcal{T}} \cup DI\_Sk(A \sqcap D) \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq (\neg(A \sqcap D) \sqcup C)$. Thus $\mathbf{T}(A \sqcap D) \sqsubseteq C$ is in the skeptical closure of $\mathcal{T}$.

Although we have established that the skeptical closure satisfies all the properties of a preferential consequence relation, we do not know yet whether it satisfies Rational Monotonicity or not, and determining whether the skeptical closure is a rational consequence relation still requires investigation.

We conclude this section with a comparison of the skeptical closure with the Basic and the Minimal Relevant Closure introduced in [17] by Casini et al. to overcome the inferential weakness of rational closure. In [17] it was shown that the Basic Relevant Closure is weaker than the Minimal Relevant Closure, and the second one is weaker than the lexicographic closure for $\mathcal{ALC}$. In [35] it was show that, in the propositional case, the Minimal Relevant Closure is weaker that the MP-closure of a knowledge base. In the following, we show that the skeptical closure is neither weaker nor stronger than the Basic (and the Minimal) Relevant closure.

A defeasible knowledge base in [17] is a pair $K = \langle \mathcal{T}, \mathcal{D} \rangle$ where the DBox $\mathcal{D}$ is a set of defeasible subsumptions $D \precsim C$ (corresponding to our inclusions $\mathbf{T}(D) \sqsubseteq C$), and the TBox $\mathcal{T}$, contains strict inclusions. When evaluating a query $C \precsim D$, one has to compute the C-justifications w.r.t. K, that is, the minimal sets of defaults $\mathcal{J} \subseteq \mathcal{D}$ making C exceptional (or, supporting $\neg C$). The idea is that, for each C-justification $\mathcal{J}$, some defeasible subsumption occurring in $\mathcal{J}$ is to be removed from $\mathcal{D}$ for consistency with C, and it is convenient to remove first the subsumptions with lower ranks.

The Relevant Closure algorithm, for a query $C \precsim D$, receives in input the ranking in the rational closure of the defeasible subsumptions in $\mathcal{D}$, and a set R of the defeasible subsumptions which are *relevant* to the query, i.e., the set of the defeasible subsumptions which are eligible for removal. The algorithm determines from $\mathcal{D}$ a new set of defeasible subsumptions $\mathcal{D}'$, by removing from $\mathcal{D}$, rank by rank, starting from the lowest rank 0, all the subsumptions in R with that rank, until the remaining set of (non-removed) defeasible subsumptions $\mathcal{D}'$ is consistent with $\mathcal{T}$ and with C.

In the Basic Relevant closure, the set R of relevant defeasible subsumptions is the union $\bigcup \mathcal{J}_j$ of all the C-justifications $\mathcal{J}_j$ w.r.t. K, where a *C-justification* $\mathcal{J}$ wrt. K is an inclusion-minimal subset of $\mathcal{D}$ such that $\top \sqsubseteq \neg C$ is in the preferential entailment of $\langle \mathcal{T}, \mathcal{J} \rangle$. By Corollary 1 in [17], a *C-justification* $\mathcal{J}$ can be equivalently defined as an inclusion-minimal subset of $\mathcal{D}$ such that $\mathcal{T} \models \overline{\mathcal{J}} \sqsubseteq \neg C$ ($\overline{\mathcal{J}}$ being the materialization of the defeasible subsumptions in $\mathcal{J}$ and $\models$ entailment in $\mathcal{ALC}$). The output $\mathcal{D}'$ of the algorithm is used, together with $\mathcal{T}$, to check whether or not $C \precsim D$ follows from $\mathcal{D}'$ and $\mathcal{T}$, i.e., whether $\mathcal{T} \models \overline{\mathcal{D}}' \sqcap C \sqsubseteq D$ (where set $\overline{\mathcal{D}}'$ is interpreted as a conjunction).

The Minimal Relevant closure exploits the same algorithm as the basic Relevant closure, but it takes $\bigcup \mathcal{J}_j^{min}$, the union of all sets $\mathcal{J}_j^{min}$, each one containing the subsumptions with lowest rank in the C-justification $\mathcal{J}_j$, as the set R of relevant defaults which are eligible for removal.

To see the difference between the skeptical closure and the Basic Relevant Closure, let us reconsider Example 3.9.

Example 6.5 *Let $K = \langle \mathcal{T}, \mathcal{D} \rangle$ be a defeasible knowledge base in [17], where $\mathcal{D}$ is the set:*

1. $Student \precsim Young$

2. $Student \precsim \neg PayTaxes$

3. $Employee \mathrel{|\!\approx} PayTaxes$
and $\mathcal{T} = \emptyset$. There is one justification of the exceptionality of $Student \sqcap Employee$ w.r.t. K, $\mathcal{J}_1 = \{2, 3\}$, therefore, the set $R = \bigcup \mathcal{J}_j = \{2, 3\}$ is used in the basic relevant closure algorithm. It contains only conditionals with rank 0, which are all removed as responsible of the exceptionality of $Student \sqcap Employee$ at the first iteration step (for rank 0). The set of remaining conditionals is $\mathcal{D}' = \{1\} = \{Student \mathrel{|\!\approx} Young\}$, so that $\overline{\mathcal{D}}' = \{\neg Student \sqcup Young\}$. Therefore, $\mathcal{T} \models \overline{\mathcal{D}}' \sqcap (Student \sqcap Employee) \sqsubseteq Young$ holds, and the subsumption $Student \sqcap Employee \mathrel{|\!\approx} Young$ is in the Basic (as well as in the Minimal) Relevant Closure of K. Instead, we have seen in Example 3.9 that the typicality inclusion $\mathbf{T}(Student \sqcap Employee) \sqsubseteq Young$ is not in the skeptical closure.

It can be seen that in Examples 5.6 and 5.9 the relevant closure behaves as the MP-closure and, in the second example, it is more cautious than the lexicographic closure. The next example shows that the skeptical closure is not weaker than the Relevant closure, let us consider the following example.

Example 6.6 *Let $\mathcal{T}$ contain the strict inclusions $Ostrich \sqsubseteq Bird$, $BabyOstrich \sqsubseteq Ostrich$, $Walk\text{-}\ Slow \sqsubseteq \neg RunFast$, and the defeasible inclusions:*

1. $\mathbf{T}(Bird) \sqsubseteq Fly$ — *2. $\mathbf{T}(Bird) \sqsubseteq WalkSlow$*
3. $\mathbf{T}(Ostrich) \sqsubseteq \neg Fly$ — *4. $\mathbf{T}(Ostrich) \sqsubseteq RunFast$*
5. $\mathbf{T}(BabyOstrich) \sqsubseteq \neg RunFast$.

We expect that typical baby ostriches inherit the defeasible property of ostriches that they do not fly, although the defeasible property $RunFast$ is overridden for typical baby ostriches. What about the property of walking slow? In the skeptical closure, one can conclude that normally baby ostriches walk slow. In RC: $rank(Bird) = 0$, $rank(Ostrich) = 1$, $rank(BabyOstrich) = 2$. Then inclusions 1 and 2 have rank 0; inclusions 3 and 4 have rank 1; inclusion 5 has rank 2. For $B = BabyOstrich$, we have $S^{sk,B} = E_2 \cup S_1^B \cup S_0^B = Strict_{\mathcal{T}} \cup \{2, 3, 5\}$, where $E_2 = Strict_{\mathcal{T}} \cup \{5\}$, $S_1^B = \{3\}$, and $S_0^B = \{2\}$. Inclusion 4 is overridden by 5, and 1 is overridden by 3. The query $\mathbf{T}(BabyOstrich) \sqsubseteq WalkSlow$ is in the skeptical closure of TBox $\mathcal{T}$, as $S^{sk,B} \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg BabyOstrich \sqcup WalkSlow$. This conclusion can be obtained from the MP-closure as well, as $S = \{2, 3, 5\}$ is the unique maximal set of defeasible inclusions compatible with B in $\mathcal{T}$.

The Relevant closure, instead, does not conclude that normally baby ostriches walk slow. Let $K' = \langle \mathcal{T}, \mathcal{D} \rangle$ be the corresponding defeasible knowledge base, where $\mathcal{T} = Strict_{\mathcal{T}}$ and $\mathcal{D} = \{1, 2, 3, 4, 5\}$ is the set of the corresponding defeasible inclusions (properly rewritten as: $Bird \mathrel{|\!\approx} Fly$, etc.). For $B = BabyOstrich$ there are three B-justifications wrt. K', namely, $\mathcal{J}_1 = \{1, 3\}$, $\mathcal{J}_2 = \{4, 5\}$ and $\mathcal{J}_3 = \{2, 4\}$, and $R = \bigcup \mathcal{J}_i = \{1, 2, 3, 4, 5\}$. The Basic Relevant Closure algorithm first removes all the defeasible subsumptions with rank 0 (i.e., 1 and 2), then the defeasible subsumptions with rank 1 in R are removed (i.e., 3 and 4), and then stops. As a result $\mathcal{D}' = \{5\}$, and $\overline{\mathcal{D}'} = \{\neg BabyOstrich \sqcup \neg RunFast\}$. Then, neither $\mathcal{T} \models \tilde{\mathcal{D}}' \sqcap BabyOstrich \sqsubseteq WalkSlow$ nor $\mathcal{T} \models \tilde{\mathcal{D}}' \sqcap BabyOstrich \sqsubseteq \neg Fly$ hold in $\mathcal{ALC}$, i.e. it does neither follow that normally baby ostriches walk slow, nor that normally baby ostriches do not fly. In the Minimal Relevant closure, $\mathcal{J}_1^{min} = \{1\}$, $\mathcal{J}_2^{min} = \{4\}$ and $\mathcal{J}_3^{min} = \{2\}$. Hence, $R = \bigcup \mathcal{J}_i^{min} = \{1, 2, 4\}$. After removing the *subsumptions with rank 0 and then with rank 1, the resulting set of defeasible subsumptions is $\mathcal{D}' = \{3, 5\}$. Then, it follows that normally baby ostriches do not fly, but not that normally baby ostriches walk slow (differently from the skeptical closure).*

Concerning the KLM properties of the Relevant Closure, it was shown in [17] that both the Basic and the Minimal Relevant Closure satisfy the property of a preferential consequence relation except *(OR)* and *(CM)*. We have seen in Proposition 6.3 that the skeptical closure instead satisfies all the properties of a preferential consequence relation.

7 Conclusions and related work

We have introduced the skeptical closure, a weak variant of the lexicographic closure [51, 23], which deals with the problem of "all or nothing" affecting the rational closure without generating alternative "bases". After the rational closure of the knowledge base is computed, checking whether a query $\mathbf{T}(C) \sqsubseteq D$ belongs to the skeptical closure requires a polynomial number of calls (in the size of the TBox) to the underlying preferential $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ reasoner, and is a problem in EXPTIME, as entailment in $\mathcal{ALC}$.

Although it is a weak construction, the skeptical closure appears to be sufficiently well-behaved as it satisfies all the KLM properties of a preferential consequence relation. We have seen that it is neither weaker nor stronger than the Basic and the Minimal Relevant Closure introduced by Casini et al. in [17]. Instead, we have proved it is weaker than the MP-closure which (as the lexicographic closure and the relevant closure), in general, requires alternative bases to be computed to check entailment of a defeasible subsumption. The models characterizing the MP-closure for $\mathcal{ALC}$ (the minimal canonical BP-models) are used to provide a semantic characterization of the skeptical closure, which has been then exploited to show that the skeptical closure satisfies the KLM properties of a preferential consequence relation.

Even if the skeptical closure requires a single basis to be computed for each query, an experimental evaluation of the approach is needed to verify whether reasoning with the skeptical closure is feasible in practice and how it compares wrt. the other refinements of the rational closure.

The logic $\mathcal{DL}^N$, introduced by Bonatti et al. in [8, 11], also deals with the problem of inheritance blocking, and builds a single extension of the knowledge base. It captures a form of "inheritance with overriding": a defeasible inclusion is inherited by a more specific class if it is not overridden by more specific (conflicting) properties. As we have seen in Example 3.9, the skeptical closure behaves differently from $\mathcal{DL}^N$, as in $\mathcal{DL}^N$ concept $WStudent$ has an inconsistent prototype: working students inherit two conflicting properties by superclasses, the property of students of not paying taxes and the property of workers of paying taxes. In the skeptical closure one cannot conclude that $\mathbf{T}(WStudent) \sqsubseteq \bot$ and, using the terminology in [8], the conflict is "silently removed". In this respect, the skeptical closure appears to be weaker than $\mathcal{DL}^N$, although it shares with $\mathcal{DL}^N$ (and with the lexicographic closure) a notion of overriding. In [11] it was shown that $\mathcal{DL}^N$ satisfies the KLM properties for *N-free* knowledge bases (when the normality operator only occurs in the l.h.s. of inclusions), a restriction which also holds here for typicality inclusions.

Another refinement of the rational closure, which deals with inheritance blocking, is the inheritance-based rational closure in [22, 24], a construction which combines the rational closure with defeasible inheritance networks. Inheritance-based rational closure, in Example 3.9, is able to conclude that typical working students are young, relying on the fact that only the information related to the connection of $WStudent$ and $Young$ (and, in particular, only the defeasible inclusions occurring on the routes connecting $WStudent$ and $Young$ in the corresponding net) are used in the rational closure construction for answering the query.

The idea of having different preference relations was first exploited by Gil [29] to define a multi-typicality formulation of the preferential logic $\mathcal{ALC}+\mathbf{T}_{min}$ [39], a logic with a preferential but not a ranked minimal model semantics. As a further difference, here we consider a single typicality operator. An extension of DLs with defeasible roles and defeasible role subsumptions has been studied by Britz and Varzinczak in [15, 13], in which multiple preference relations associated with roles are considered.

Another related approach by Bozzato et al. in [12] develops an extension of the CKR framework in which defeasible axioms are allowed in the global context and can be overridden by knowledge in a local context. Exceptions have to be justified in terms of semantic consequence. A translation of extended CHRs (with knowledge bases in $\mathcal{SROIQ}$-RL) into Datalog programs under the answer set semantics is also defined.

Concerning the multipreference semantics introduced in [44] (and further refined in [43]) to provide a strengthening of the rational closure, we have shown in [43, 32] that the MP-closure is a sound construction for the multipreference semantics. Here, we have given a semantic characterization of the MP-closure for $\mathcal{ALC}$ in terms of minimal canonical BP-models, a semantics that, as a consequence, is weaker than the multipreference semantics.

The relationships among the above variants of rational closure and the notions of rational closure defined for DLs in the contexts of fuzzy logic [20] and probabilistic logics [52, 60] have not been investigated so far. In the propositional logic case, it has been shown in [4] that the KLM preferential logics and the rational closure [49, 50], the probabilistic approach [1], the system Z [56] and the possibilistic approach [5, 4] are all related with each other, and similar relations might be expected to hold for non-monotonic extensions of description logics as well. The relationships with c-representations [47], which are able to handle forms of irrelevance and inheritance in the propositional case, are to be investigated as well.

As the definitions of rational closure for expressive DLs [36], for low complexity DLs [57, 58, 36, 25], and for all DLs [7] have been recently investigated, a natural question arising is whether the skeptical closure and other closure constructions defined for $\mathcal{ALC}$ can be extended to these DLs as well. This investigation is left for future work.

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